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Research on the Application of Vector Algebra in Elementary Geometry

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***Abstract*—Vector is an important concept in mathematics and serves as a bridge between algebra and geometry. Vector algebra has important practical application value This article mainly explores the application of vectors in elementary geometry, including using vectors to determine collinearity, coplanar, and coplanar points, calculating distances, angles, areas, and volumes, and proving perpendicularity.**

***Index Terms*—Elementary geometry; Standard frame; Coordinates.**

I. Introduction

In multiple disciplines such as mechanics, physics, and engineering technology, vectors are endowed with many practical meanings, such as displacement, velocity, acceleration, moment of inertia, etc. [1]. In elementary mathematics, we put aside these practical meanings and use directed line segments to represent vectors more vividly

Many geometric problems in elementary mathematics are difficult to prove or calculate and have strong skills. Some problems even need to add more auxiliary lines, which brings great difficulties to solve problems [2-8]. However, using vector methods will often make the solution much easier We introduce vectors into elementary geometry and transform the problems we consider into vector operations to approach these problems from another perspective. This article mainly explores the application of vectors in elementary geometry, including using vectors to determine collinearity, coplanarity, and collinearity, calculating distance, angle, area, and volume, and proving perpendicularity.

II. The Application of Vectors in Elementary Geometry

A. Using vectors to determine point collinearity and coplanarity

Using vectors to determine whether points are collinear or coplanar can simplify the problem.

Example 2.1 Given any two non collinear non zero vector $\vec{a}$ and $\vec{b}$ in the space, make

$$\overrightarrow{AB}=\vec{a}+\vec{b},\ \overrightarrow{BC}=2\vec{a}+8\vec{b},\ \overrightarrow{CD}=3(\vec{a}-\vec{b}),$$

and try to judge the position relationship between A, B and D.

Analysis. To determine the positional relationship between three points, the main thing is to see if they are collinear. Since two points determine a straight line, if the third point can be determined to be on this line, then these three points can be determined to be collinear. In this problem, we can use vector knowledge to determine whether the three points are collinear, that is, by determining whether the vectors $\overrightarrow{AD}$ and $\overrightarrow{AB}$ are collinear, that is, by determining whether there is λ, $\overrightarrow{AD}=\lambda\overrightarrow{AB}$ is established.

Solution. Based on the known conditions of the question, we can be obtain

$$\begin{aligned}\overrightarrow{AD}&=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}\\&=(\vec{a}+\vec{b})+(2\vec{a}+8\vec{b})+3(\vec{a}-\vec{b})\\&=6(\vec{a}+\vec{b})\\&=6\overrightarrow{AB}.\end{aligned}$$

So $\overrightarrow{AD}$ and $\overrightarrow{AB}$ are collinear and have a common point A, so A, B and D are collinear.

Example 2.2 Given four points $A(1,0,1)$, $B(4,4,6)$, $C(2,2,3)$, $D(10,14,17)$ in space, prove that A, B, C, D are coplanar at four points.

Proof. From known conditions,we obtain

$$\overrightarrow{AB}=(3,4,5),\ \overrightarrow{AC}=(1,2,2),\ \overrightarrow{AD}=(9,14,16).$$

Let $\overrightarrow{AD}=x\overrightarrow{AB}+y\overrightarrow{AC}(x,y\in R)$, then

$$(9,14,16)=x(3,4,5)+y(1,2,2).$$

So there is

$$\begin{cases}3x+y=9\\4x+2y=14\\5x+2y=16\end{cases}\Rightarrow\begin{cases}x=2\\y=3.\end{cases}$$

Therefore, $\overrightarrow{AD}=2\overrightarrow{AB}+3\overrightarrow{AC}$.

Thus, A, B, C and D are coplanar.

B. Using vectors to determine line collinearity

For some problems of determining line collinearity, it is often difficult to prove using general geometric methods, so we use vector methods to solve such problems and simplify the proof process.

Example 2.3 Proves that the lines connecting the midpoints of the opposite sides of a tetrahedron intersect at a point and are evenly divided.

Proof. As shown in Figure 1, in tetrahedral $ABCD$, let the midpoints of one set of opposite edges AB and CD be E, F,

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connect E, F, and let the midpoint of EF be P_1. Similarly, let the midpoints of the other two sets of opposite edges midpoint connections be P_2 and P_3, respectively. Below, all we need to prove is that P_1, P_2 and P_3 coincide.

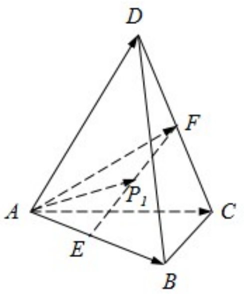

Figure 1

Take three non coplanar vectors $\overrightarrow{AB}=\vec{e_1}$, $\overrightarrow{AC}=\vec{e_2}$, $\overrightarrow{AD}=\vec{e_3}$, and first find the relationship of $\overrightarrow{AP_1}$ expressed linearly by $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$.

Connect A and F, because AP_1 is the centerline of the triangle *AEF*, so we have $\overrightarrow{AP_1}=\frac{1}{2}(\overrightarrow{AE}+\overrightarrow{AF})$.

Because *AF* is the centerline of the triangle *ACD*, there is also $\overrightarrow{AF}=\frac{1}{2}(\overrightarrow{AC}+\overrightarrow{AD})=\frac{1}{2}(\vec{e_2}+\vec{e_3})$.

Since $\overrightarrow{AE}=\frac{1}{2}\overrightarrow{AB}=\frac{1}{2}\vec{e_1}$, so

$$\overrightarrow{AP_1}=\frac{1}{2}\left[\frac{1}{2}\vec{e_1}+\frac{1}{2}(\vec{e_2}+\vec{e_3})\right]$$
$$=\frac{1}{4}(\vec{e_1}+\vec{e_2}+\vec{e_3}).$$

Similarly, it can be concluded that

$$\overrightarrow{AP_i}=\frac{1}{4}(\vec{e_1}+\vec{e_2}+\vec{e_3})\ \ (i=2,3).$$

So there is $\overrightarrow{AP_1}=\overrightarrow{AP_2}=\overrightarrow{AP_3}$.

Thus, it is known that P_1, P_2 and P_3 coincide, and the proposition is proven.

C. Using vectors to determine the distance between a point and a plane

When we solve solid geometry problems, we often need to have a strong spatial thinking ability, which will increase the difficulty of solving problems. If we can solve these problems with calculation and turn geometric problems into algebraic problems, it will greatly reduce the difficulty and improve the ability to solve problems. At this time, vector algebra is very important.

Example 2.4 As shown in Figure 2, in the straight triangular prism *ABC-A₁B₁C₁*, its bottom triangle *ABC* is an isosceles right triangle, $\angle ACB=90°$, side edges $AA_1=2$, D and E are the midpoints of CC_1 and A_1B respectively, and the projection of point E on the plane ABD is the center of gravity G of triangle ΔABD.

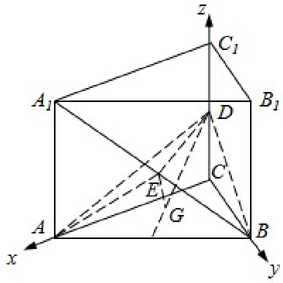

Figure 2

Solution. With *C* as the origin, *CA, CB* and CC_1 as the *x*, *y* and *z* axes respectively, establish a spatial Cartesian coordinate system coordinate system. Set $AC=2a$, then we get

$A(2a,0,0)$, $B(0,2a,0)$, $D(0,0,1)$,

$A_1(2a,0,2)$, $E(a,a,1)$, $G(\frac{2}{3}a,\frac{2}{3}a,\frac{1}{3})$.

So we have $\overrightarrow{GE}=(\frac{a}{3},\frac{a}{3},\frac{2}{3})$, $\overrightarrow{BD}=(0,-2a,1)$.

Because the projection of point *E* on plane *ABD* is the center of gravity *G* of ΔABD, so $\overrightarrow{GE}\cdot\overrightarrow{BD}=-\frac{2}{3}a^2+\frac{2}{3}=0$. Thus we have $a=1$. Therefore, there are

$\overrightarrow{AD}=(-2,0,1)$, $\overrightarrow{DE}=(1,1,0)$, $\overrightarrow{AA_1}=(0,0,2)$.

Assuming a normal vector of plane *ADE* is *ADE*, then

$$\begin{cases}\vec{n}\cdot\overrightarrow{AD}=0\\ \vec{n}\cdot\overrightarrow{DE}=0\end{cases}\Rightarrow\begin{cases}-2x+1=0\\ x+y=0.\end{cases}$$

Solving this system of equations yields

$$x=\frac{1}{2},\ y=-\frac{1}{2}.$$

Hence, we get

$$\vec{n}=(\frac{1}{2},-\frac{1}{2},1).$$

So the distance from point A_1 to plane ADE is

$$d=\frac{|\vec{n}\cdot\overrightarrow{AA_1}|}{|\vec{n}|}=\frac{2\sqrt{6}}{3}.$$

D. Using vectors to calculate area and volume

By utilizing the mixed product properties of two vectors and three vectors, the area of a triangle and the volume of a tetrahedron can be calculated.

Example 2.5 Given three points *A*(1,2,3), *B*(2,-1,5), C(3,2,-5) in space, find the area of triangle ΔABC.

Solution. The area of triangle *ABC* is

$$S_{\Delta ABC}=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$$

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Since $\overrightarrow{AB}=(1,-3,2)$, $\overrightarrow{AC}=(2,0,-8)$,

so we have $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -3 & 2\\ 2 & 0 & -8\end{vmatrix}=24\vec{i}+12\vec{j}+6\vec{k}.$

Therefore, we get $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{24^2+12^2+6^2}=6\sqrt{21}.$

Hence, the area is $3\sqrt{21}$.

In elementary geometry, some books or literatures use the method of adding auxiliary lines and using geometric relations to solve some problems of finding the volume of geometry [1-8]. This method requires strong spatial imagination and is not easy to solve problems. Therefore, here we can use the method of establishing a spatial Cartesian coordinate system system, using the normal vector of the plane to find the height of the tetrahedron, as well as the volume of the geometry, The method is more concise.

Example 2.6 It is known that E and F are the midpoints of the edges BC and FF of the cube $ADCE$. If the edge length of the cube is 1, calculate the volume of the tetrahedral $ACDR$.

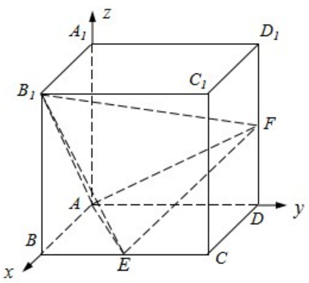

Figure 3

Solution. As shown in Figure 3, take A as the origin, and the straight lines of AB, AD, and AA_1 are the x, y, and z axes respectively, to establish a spatial Cartesian coordinate system system, then $A(0,0,0)$, $B_1(1,0,1)$, $E(1,\frac{1}{2},0)$, $F(0,1,\frac{1}{2})$.

So we have $\overrightarrow{AE}=(1,\frac{1}{2},0)$, $\overrightarrow{AF}=(0,1,\frac{1}{2})$, $\overrightarrow{EF}=(-1,\frac{1}{2},\frac{1}{2})$.

Therefore the triangle AEF is a isosceles triangle.

Let the height on EF is h, then $h=\sqrt{AE^2-(\frac{1}{2}EF)^2}=\frac{\sqrt{14}}{4}$.

So we get $S_{\Delta AEF}=\frac{1}{2}\cdot EF\cdot h=\frac{1}{2}\cdot\frac{\sqrt{6}}{2}\cdot\frac{\sqrt{14}}{4}=\frac{\sqrt{21}}{8}$.

Assuming the distance from point B_1 to plane AEF is d and the normal vector of plane AEF is $\vec{n}=(x,y,z)$, then

$$\begin{cases}\vec{n}\cdot\overrightarrow{AE}=0\\ \vec{n}\cdot\overrightarrow{AF}=0\end{cases}\Rightarrow\begin{cases}x+\frac{1}{2}y=0\\ y+\frac{1}{2}z=0\end{cases}\Rightarrow\begin{cases}x=-\frac{1}{2}y\\ z=-2y.\end{cases}$$

Take $\vec{n}=(-1,2,-4)$, $\overrightarrow{B_1F}=(-1,1,-\frac{1}{2})$, then

$$d=\frac{|\overrightarrow{B_1F}\cdot\vec{n}|}{|\vec{n}|}=\frac{5}{\sqrt{21}}.$$

Therefore, the volume of tetrahedral $A-B_1EF$ is

$$V=\frac{1}{3}S_{\Delta AEF}\cdot d=\frac{1}{3}\cdot\frac{\sqrt{21}}{8}\cdot\frac{5}{\sqrt{21}}=\frac{5}{24}.$$

E. Using vectors to calculate the angle between a line and a plane

We know that in elementary geometry, the cosine value of the angle between a line and a plane normal vector is equal to the sine value of the angle between the line and the plane. Therefore, the cosine formula for calculating the angle between two vectors can be used to conveniently calculate the angle.

Example 2.7 As shown in Figure 4, in the straight triangular prism $ABC-A_1B_1C_1$, if

$CA\perp CB$, $CA=CB=1$, $AA_1=2$,

both M and N are the midpoints of A_1B_1 and A_1A respectively, calculate the sine value of the angle between the straight line B_1C and the plane C_1MN.

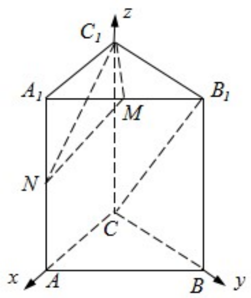

Figure 4

Solution. Take C as the origin, and the straight lines of CA, CB, and CC_1 are the x, y, and z axes respectively. Establish a spatial Cartesian coordinate system system, then we can get

$C(0,0,0)$, $B_1(0,1,2)$, $C_1(0,0,2)$,

$M(\frac{1}{2},\frac{1}{2},2)$, $N(1,0,1)$.

So we have

$\overrightarrow{C_1M}=(\frac{1}{2},\frac{1}{2},0)$, $\overrightarrow{C_1N}=(1,0,-1)$, $\overrightarrow{CB_1}=(0,1,2)$.

Assuming that one normal vector of the planar C_1MN is $\vec{n}=(x,y,z)$, then

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$$\begin{cases} \vec{n}\cdot\overrightarrow{C_1M}=0 \\ \vec{n}\cdot\overrightarrow{C_1N}=0 \end{cases} \Rightarrow \begin{cases} \frac{1}{2}x+\frac{1}{2}y=0 \\ x-z=0 \end{cases} \Rightarrow x=-y=z.$$

Taking $\vec{n}=(1,-1,1)$, then

$$\cos\angle(\vec{n}\cdot\overrightarrow{CB_1})=\frac{\vec{n}\cdot\overrightarrow{CB_1}}{|\vec{n}|\cdot|\overrightarrow{CB_1}|}=\frac{\sqrt{15}}{15}.$$

Because the cosine value of the angle between the line and the plane normal vector is equal to the sine value of the angle between the line and the plane, the sine value of the angle between the line B_1C and the plane C_1MN is $\frac{\sqrt{15}}{15}$.

F. Using vectors to prove vertical problems

Geometric proof is often an important and challenging part of elementary geometry. When solving such problems, some of them are either abstract, highly technical, or require the addition of many auxiliary lines, making it difficult for us to solve them. If we introduce vectors in geometric proof, the problem becomes much simpler. [9] examined the development and refinement of possible mathematical models for the intellectual system of career guidance. Mathematical modeling of knowledge expression in the career guidance system, Combined method of eliminating uncertainties, Chris-Naylor method in the expert information system of career guidance, Shortliff and Buchanan model in the expert information system of career guidance and DempsterSchafer in the expert information system of career guidance method has been studied. The algorithms of the above methods have been developed. The set of hypotheses in the expert system is the basic structure of the system that determines the set of possible decisions of the expert system. This set, which is crucial in decision-making, should be sufficiently complete to describe all the possible consequences of situations that arise in the subject area. Therefore, it is important to improve the mathematical models of the intellectual system of career guidance. [10] discussed about specific Policy document which ensures of which the teaching, learning in addition to assessment methods are upwards to the amount of typically the course and are ideal to the attainment involving objectives and intended understanding outcomes of the program and the course. The particular policy requires that school members use recent in addition to variety of teaching, mastering methods and assessment methods. Higher Quality Accredited Institutions will continue to further more improve the standard involving teaching and learning via recognition, sharing and moving of good practices to be able to inspire the learners to be able to achieve their potentials throughout a multicultural environment in addition to in turn, improve accomplishment, retention and learners pleasure

Below are two examples of using vector method to prove vertical problems

Example 2.8 Prove that if the sum of squares of one set of opposite sides of a quadrilateral is equal to the sum of squares of another set of opposite sides, then its diagonals are perpendicular to each other.

Analysis. If we use general geometric methods to prove this problem, we need to divide it into three situations to discuss. One is that the quadrilateral is convex quadrilateral, the second is concave quadrilateral, and even more difficult is the situation when the quadrilateral is folded quadrilateral [8]. In elementary geometry, if we use this method to prove, it is too cumbersome, but if we introduce vectors, we do not need to discuss these, This question will become much simpler. Let's use vector method to prove it.

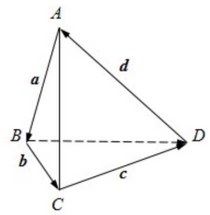

Figure 5

Proof. As shown in Figure 5, let $\overrightarrow{AB}=\vec{a}$, $\overrightarrow{BC}=\vec{b}$, $\overrightarrow{CD}=\vec{c}$, $\overrightarrow{DA}=\vec{d}$ and $|\vec{a}|=a$, $|\vec{b}|=b$, $|\vec{c}|=c$, $|\vec{d}|=d$ in a quadrilateral $ABCD$. If $a^2+c^2=b^2+d^2$ is known, it is necessary to prove that $\overrightarrow{AC}\perp\overrightarrow{BD}$.

Because $\vec{a}+\vec{b}+\vec{c}+\vec{d}=0$, so $\vec{a}+\vec{b}=-\vec{c}-\vec{d}$.

Take the square of both sides to obtain

$$\vec{a}^2+\vec{b}^2+2\vec{a}\vec{b}=\vec{c}^2+\vec{d}^2+2\vec{c}\vec{d}.$$

And because

$$\vec{a}^2-\vec{b}^2+2\vec{b}^2+2\vec{a}\vec{b}=\vec{d}^2-\vec{c}^2+2\vec{c}^2+2\vec{c}\vec{d},$$

So we have

$$\vec{a}^2-\vec{b}^2+2\vec{b}(\vec{a}+\vec{b})=\vec{d}^2-\vec{c}^2+2\vec{c}(\vec{c}+\vec{d}).$$

Hence

$$a^2-b^2+2\vec{b}\cdot\overrightarrow{AC}=d^2-c^2+2\vec{c}\cdot\overrightarrow{CA}.$$

Therefore, we obtain

$$2\vec{b}\cdot\overrightarrow{AC}=2\vec{c}\cdot\overrightarrow{CA}.$$

So we get $2\overrightarrow{AC}(\vec{b}+\vec{c})=0$, i.e., $2\overrightarrow{AC}\cdot\overrightarrow{BD}=0$.

III. CONCLUSION

As an important bridge connecting algebra and geometry, vectors have always been present in all branches of mathematics. Introducing vectors into elementary geometry and using vector algebra to solve problems, the basic idea is to utilize the special properties of both algebraic and geometric forms of vectors, which can combine numbers and shapes, connect geometry and algebra, quantify and algebraize spatial structures, and transform abstract qualitative problems into simpler quantitative problems

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In this paper, aiming at some problems in elementary geometry, by establishing a space Cartesian coordinate system system, we can make the points on the coordinate system, that is, ordered real number pairs, correspond to each other one by one, quantify geometric problems, and help us solve problems more simply.

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REFERENCES

[1] Q.H. Sun. Historical Research on Vector Theory. Northwestern University, 2006:7-123.
[2] H.F Dai. Historical Research on vector space Theory. Shandong University, 2013:4-49.
[3] Morris Kline.Mathematical Thought from Ancient to Modern Times.Oxford University Press，1990.
[4] L.H. Zeng, H.M. Duan, L. Li. Advanced Algebra and Analytic Geometry. Beijing: Tsinghua University Press, 2014.
[5] L. G. Lv, Z.D. Xu. Analytic Geometry. 4th Edition. Beijing: Higher Education Press, 2006.
[6] Y.J. Sang. Application of vector algebra in elementary mathematics. Journal of Xinjiang Workers' University, 1998 (4): 61-63.
[7] C.F. Yang. The Role of Vectors in Analytic Geometry. Journal of Yuxi University (Natural Science Edition), 2005, 4 (1): 4-7.
[8] T.H. Wang. On the Application of Vector Algebra to trigonometry [J]. Science and Technology Innovation Forum, 2011 (4): 185-190.
[9] Christo Ananth, A.R. Akhatov, D.R. Mardonov, F.M. Nazarov, T. AnanthKumar, "Possible Models and Algorithms for the Intellectual System of Professional Direction", International Journal of Early Childhood Special Education, Volume 14, Issue 05, 2022,pp. 4133-4145.
[10] Christo Ananth, "Policy Document on Teaching, Learning and Assessment in Higher Quality Accredited Institutions", International Journal of Advanced Research Trends in Engineering and Technology (IJARTET),Volume 5,Issue 2,February 2018,pp:27-31

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