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The Application of Convex Functions in the Proof of Inequalities

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***Abstract*—Convex functions are a class of commonly seen functions with good properties, which have been widely applied to practical problems in mathematics and many other fields. This article mainly introduces the application of convex functions in proving basic inequalities, integral inequalities, trigonometric function inequalities, and other aspects through examples.**

***Index Terms*—Convex function, inequality, integral inequality, trigonometric function inequality.**

I. INTRODUCTION

The concept of convex functions was first reflected in the theory proposed by Danish mathematician Jensen, and also appeared in the theory of another mathematician Adama. Jensen's inequality is generally used to prove non infinite inequalities, It is an important link connecting the sum of infinite terms and the integration. If the geometric and arithmetic mean of two non-negative numbers are to be refined, it needs to be calculated through the Adama inequality. In numerous works both at home and abroad, the study of convex functions has gradually shifted from preliminary definitions and properties to convex analysis and the study of multivariate convex functions, and the exploration of convex functions has gradually deepened, Convex functions have become an important research direction in mathematics [1-6].

The convexity of a function can be used to find the inflection points, maximum points, and stable points of the function, as well as to solve convex optimization problems. This article mainly introduces the application of convex functions in proving basic inequalities, integral inequalities, trigonometric function inequalities, and other aspects through examples. [7] examined the development and refinement of possible mathematical models for the intellectual system of career guidance. Mathematical modeling of knowledge expression in the career guidance system, Combined method of eliminating uncertainties, Chris-Naylor method in the expert information system of career guidance, Shortliff and Buchanan model in the expert information system of career guidance and DempsterSchafer in the expert information system of career guidance method has been studied. [8] discussed that according to the observations in this paper, an existing mathematical model of banking capital dynamics should be tweaked. First-order ordinary differential equations with a "predator-pray" structure make up the model, and the indicators are competitive. Numerical realisations of the model are required to account for three distinct sets of initial parameter values. It is demonstrated that a wide range of banking capital dynamics can be produced by altering the starting parameters. One of the three options is selected, and the other two are eliminated.

II. APPLICATION OF CONVEX FUNCTIONS

Convex functions are a special type of function that has a wide range of applications in the field of mathematics. When proving inequalities, certain special properties of convex functions are often utilized Some unique properties of convex functions can transform tedious inequalities to obtain the basic form of convex functions, and thus draw conclusions. Below, we will provide examples to illustrate the application of convex functions in the proof of inequalities.

A. The Application of Convex Functions in the Proof of Basic Inequalities

Example 1. Proves the inequality

$$\frac{1}{2}(a^n+b^n)>\left(\frac{a+b}{2}\right)^n,$$

where a and b are numbers greater than zero and are not equal, $n>1$.

Proof. The use of construction methods to obtain suitable convex functions and utilize their properties to solve problems.

Assuming $h(t)=t^n\ (t>0,n>1)$, perform first-order differentiation on this function to obtain $h'(t)=nt^{n-1}$, and then perform second-order differentiation on it to obtain

$$h''(t)=n(n-1)t^{n-2}\ (t>0,n>1).$$

According to the theorem for determining convex functions, $h(t)$ is a strictly convex function on $t>0$, and the definition of convex functions yields

$$h\left(\frac{t_1+t_2}{2}\right)\le\frac{h(t_1)+h(t_2)}{2},$$

Therefore, we obtain

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$$\frac{1}{2}\left(a^n+b^n\right)>\left(\frac{a+b}{2}\right)^n.$$

Example 2. Proves that $e^{\frac{a+b}{2}} \le \frac{1}{2}(e^a+e^b)$.

Proof. Assuming $h=e^x$, by taking the second derivative of the function and judging its sign, it is concluded that the function is a convex function in the defined domain.

Use m to represent a, n to represent $\frac{a+b}{2}$, and p to represent b (m,n,p are numbers within the domain).

$h=e^x$ is a convex function within the domain. From the properties of convex functions, it can be seen that if $\frac{h(n-m)}{n-m} \le \frac{h(p-m)}{p-m}$, the following inequality holds

$$\frac{e^{\frac{a+b}{2}}-e^b}{\frac{a+b}{2}-a} \le \frac{e^b-e^a}{b-a}.$$

Obtained after sorting

$$e^{\frac{a+b}{2}}-e^a \le \frac{1}{2}(e^b-e^a).$$

Thus we have

$$e^{\frac{a+b}{2}} \le \frac{1}{2}(e^a+e^b).$$

B. The Application of Convex Functions in the Proof of Integral Inequalities

Example 3 If f is a convex function on the interval $[0,b](b>0)$, then

$$\frac{1}{x+y}\int_0^{x+y} f(t)dt \ge \frac{1}{y-x}\int_x^y f(t)dt$$

holds for any $x,y\in[0,b]$, $x\ne y, x+y\le b$.

Proof. If $x<y, x=0$, there is obviously

$$\frac{1}{x+y}\int_0^{x+y} f(t)dt \ge \frac{1}{y-x}\int_x^y f(t)dt.$$

If $x>0$, $F(x)=\frac{1}{x}\int_0^x f(t)dt$, it is known from the properties of convex functions that F is still a convex function on $[0,b](b>0)$. However, $x<y<x+y$, so

$$xF(x)-yF(y)+(y-x)F(x+y)\ge 0,$$

i.e.,

$$\int_0^x f(t)dt-\int_0^y f(t)dt+\frac{y-x}{x+y}\int_0^{x+y} f(t)dt \ge 0.$$

Considering

$$\int_0^x f(t)dt-\int_0^y f(t)dt=-\int_x^y f(t)dt, y>x,$$

so we have

$$\frac{1}{x+y}\int_0^{x+y} f(t)dt \ge \frac{1}{y-x}\int_x^y f(t)dt.$$

Example 4. If the function $f(x)\in R[a,b]$ and $m<f(x)<M, x\in[a,b]$, $h(u)$ is a continuous convex function on the interval $[m,M]$. Prove that

$$h\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \le \frac{1}{b-a}\int_a^b h(f(x))dx.$$

Proof. Divide the closed interval $[a,b]$ n equally, with points in order of $a=x_0<x_1<\cdots<x_n=b$. Because $h(u)$ is a convex function, so we have

$$h\left(\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\right)$$
$$\le \frac{1}{n}(h(f(x_1))+h(f(x_2))+\cdots+h(f(x_n))),$$

i.e.,

$$h\left(\frac{1}{b-a}\sum_{i=1}^n f(x_i)\frac{b-a}{n}\right) \le \frac{1}{b-a}\sum_{i=1}^n h(f(x_i))\frac{b-a}{n}.$$

Since $f(x)\in R[a,b]$ and $h(u)$ is continuous convex function on the interval $[m,M]$, so $h(f(x))\in R[a,b]$. Setting $n\to\infty$, then we have

$$h\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \le \frac{1}{b-a}\int_a^b h(f(x))dx.$$

C. The Application of Convex Functions in the Proof of Triangular Function Inequalities

Example 5 In triangle ABC, prove that:

(1) $\sin A+\sin B+\sin C \le \frac{3\sqrt{3}}{2}$;

(2) $\sin A\sin B\sin C \le \frac{3\sqrt{3}}{8}$.

Proof. (1) Set $h(x)=-\sin x, x\in(0,\pi)$, then $h''(x)=\sin x>0$. Since $h(x)$ is a convex function on the domain of definition, and according to Johnson's inequality,

$$-\sin\left(\frac{A+B+C}{3}\right) \le -\frac{\sin A+\sin B+\sin C}{3},$$

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$$\frac{\sqrt{3}}{2}=\sin\left(\frac{\pi}{3}\right)\geq\sin\left(\frac{A+B+C}{3}\right)$$
$$\geq\frac{\sin A+\sin B+\sin C}{3}.$$

Therefore, we have

$$\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}.$$

(2) Assuming $f(x)=-\lg x$, , then $f(x)$ is a convex function in the domain $(0,+\infty)$, and because sin$A$, sin$B$, and sinC are all positive numbers, and $f(x)=-\lg x$, is monotonically decreasing in the domain, according to Johnson's inequality,

$$-\lg(\sin A\sin B\sin C)=-(\lg\sin A+\lg\sin B+\lg\sin C)$$
$$\geq-3\lg\frac{\sin A+\sin B+\sin C}{3}$$
$$\geq-3\lg\frac{\sqrt{3}}{2}.$$

Therefore, we have $\sin A\sin B\sin C\leq\frac{3\sqrt{3}}{8}$.

Example 6. Let a,b be any non negative real number, and prove that $2\arctan\left(\frac{a+b}{2}\right)\geq\arctan a+\arctan b$.

Proof. Assuming that $p(x)=-\arctan x\,(x>0)$, the second derivative of this function is

$$p''(x)=\frac{2x}{(1+x^2)^2}\,(x>0).$$

Obviously, the function $p''(x)$ is a strictly convex function on $(0,+\infty)$. Therefore, for any non negative real numbers a,b, we get

$$p\left(\frac{a+b}{2}\right)\leq\frac{p(a)+p(b)}{2},\text{y}$$
$$-\frac{\arctan a+\arctan b}{2}\geq-\arctan\left(\frac{a+b}{2}\right).$$

Therefore, we have

$$2\arctan\left(\frac{a+b}{2}\right)\geq\arctan a+\arctan b.$$

From the above example, it can be seen that when proving certain inequalities, using the properties of convex functions and transforming them through constructors can simplify the proof. The key lies in constructing appropriate functions.

III. Conclusion

For some inequalities that are difficult to prove, a simple proof can be provided by using convex functions. By using convex functions, these seemingly cumbersome inequality proof problems become clear and clear. When using convex functions to prove inequalities, we often first modify the original inequality by constructing it into the function we need to obtain, thus solving the problem in a clear way.

Convex functions are not only very useful in inequalities, but have been widely used in economics and even other fields, and are worth further exploration.

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