



Some Applications of Newton's Interpolation Method in Secondary School Mathematics

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Abstract—In this paper we focus on the application of Newton interpolation method in secondary mathematics. We illustrate the use of Newton interpolation method by giving examples from determining the quadratic curve equation, proving the constant equation and solving the system of linear equations, respectively. We hope that our work can provide some new ideas and approaches to solve the above-mentioned secondary mathematical problems.

Index Terms—Newton interpolation, Secondary mathematics, Solving a system of equations.

I. INTRODUCTION

Newton interpolation is an important element in numerical analysis. Knowing the values of $f(x)$ at $n+1$ points x_0, x_1, \dots, x_n on some interval $[a, b]$ that are mutually exclusive, we need to find a polynomial $P_n(x)$ that has no more than n times such that $P_n(x_j) = f(x_j) (j=0, 1, \dots, n)$ ($j=0, 1, \dots, n$). At this point, $f(x)$ is called the interpolated function, $[a, b]$ is the interpolation interval, x_0, x_1, \dots, x_n is the interpolation node or interpolation base point, $P_n(x)$ is the interpolation polynomial, and $P_n(x_j) = f(x_j) (j=0, 1, \dots, n)$ is the interpolation condition. For this problem, Newton's interpolation method can be used to find the interpolating polynomial. Newton's interpolation method, from its original idea, originated in elementary mathematics and emerged from elementary mathematics. In the following we first introduce Newton's interpolation method and present some of its applications in secondary school mathematics [1-4].

The essence of Newton interpolation is to seek a polynomial function of the n th order of an image passing through $n+1$ different points. Let these $n+1$ point be: $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$, we construct the n th order polynomial $f(x)$ as follows.

$$P_n(x) = f(x_0) + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)L \\ + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)L \dots (x-x_n)$$

Where $f[x_0, x_1, L, x_n]$ is the n th order mean difference, which is defined as follows.

Definition 1[1]: The first-order mean difference of the function $f(x)$ with respect to the points x_0, x_k is called

$$f[x_0, x_k] = \frac{f(x_k) - f(x_0)}{x_k - x_0}.$$

In general, the k th order mean difference of the function $f(x)$ with respect to the points x_0, x_1, \dots, x_k is called

$$f[x_0, x_1, L, x_k] = \frac{f[x_1, x_2, L, x_k] - f[x_0, x_1, L, x_{k-1}]}{x_k - x_0}$$

According to the definition of the mean difference, the n interpolation polynomial can be obtained by substituting $n+1$ coordinate points.

Theorem 1[1]: An interpolating polynomial satisfying $P_n(x_j) = f(x_j) (j=0, 1, \dots, n)$ exists and is unique.

If mean differences are required, a table of mean differences can be presented as follows.

TABLE I: MEAN DIFFERENCE TABLE

x_k	$f(x_k)$	First-order	Second-order	Third-order	Fourth-order
x_0	$f(x_0)$				
x_1	$f(x_1)$	$f[x_0, x_1]$			
x_2	$f(x_2)$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$		
x_3	$f(x_3)$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$	
x_4	$f(x_4)$	$f[x_3, x_4]$	$f[x_2, x_3, x_4]$	$f[x_1, x_2, x_3, x_4]$	$f[x_0, x_1, x_2, x_3, x_4]$
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II. APPLICATION OF NEWTON'S INTERPOLATION METHOD

A. Using Newton interpolation to find polynomial functions over a number of points

Example 1: Find a quadratic polynomial $f(x)$ with respect to x such that $f(2) = 4$, $f(5) = 6$, and $f(6) = 11$.

Solution: Method I (method of coefficients to be determined)

Let the quadratic function be $f(x) = ax^2 + bx + c$, according to the known conditions $f(2)=4$, $f(5)=6$, $f(6)=11$, substitute the following system of equations

$$\begin{cases} 4a + 2b + c = 4 \\ 25a + 5b + c = 6 \\ 36a + 6b + c = 11 \end{cases}$$

Solving this system of equations yields

$$a = \frac{13}{12}, b = \frac{-83}{12}, c = \frac{27}{2}$$

Therefore the equation is



$$f(x) = \frac{13}{12}x^2 - \frac{83}{12}x + \frac{27}{2}$$

Method II (Newton's interpolation method)

Let $x_0 = 2, x_1 = 5, x_2 = 6$, we construct the Newton interpolation polynomial

$$P_2(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

From the mean difference table II we can get

TABLE II: MEAN DIFFERENCE TABLE FOR EXAMPLE 1

x_k	$f(x_k)$	First-order	Second-order
2	4		
5	6	3/2	
6	11	5	13/12

$$P_2(x) = 4 + \frac{3}{2}(x-2) + \frac{13}{12}(x-2)(x-4) = \frac{13}{12}x^2 - \frac{83}{12}x + \frac{27}{2}$$

Therefore,

$$f(x) = \frac{13}{12}x^2 - \frac{83}{12}x + \frac{27}{2} \quad \#$$

Comparing the above two methods, Method 1 is a common idea, but the process of solving the system of equations is tedious; while method 2 can be obtained by listing the mean difference. This is more convenient than method 1.

B. Newton's interpolation polynomials in analytic geometry

Example 2: Find the equation of the parabola passing through the points A(0,1), B(-1,1), and C(1,-1) with the axis of symmetry parallel to the y-axis.

Solution: Since the axis of symmetry of the parabola is parallel to the y-axis, according to the relationship between the equation of the parabola and the graph, let

$$x_0 = 0, x_1 = -1, x_2 = 1,$$

$$f(x_0) = 1, f(x_1) = 1, f(x_2) = -1,$$

We construct the polynomial as follows

$$y = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

From the mean difference table we obtain

TABLE II: MEAN DIFFERENCE TABLE FOR EXAMPLE 2

x_k	$f(x_k)$	First-order	Second-order
0	1		
-1	1	0	
1	-1	-1	-1

$$y = 1 - (x-0)(x+1) = x^2 - x + 1,$$

which is the desired parabola.

Example 3: Find the equation of the ellipse passing through the point (2, 6), $(4, 2\sqrt{3} + 3)$ and tangent to the y-axis at the point (0, 3) with the axis of symmetry parallel to the coordinate axis.

Solution: According to the problem, the elliptic equation has the following form

$$\frac{(x-x_0)^2}{a^2} + \frac{(x-3)^2}{b^2} = 1$$

We will find the value of x_0, a, b as follows.

Let $f(x) = (y-3)^2$, then $f(x)$ is a quadratic polynomial function on x . We also set up

$$f(x) = a_0 + a_1x + a_2x(x-2)$$

Since $f(x)$ passes through the points (0, 0), (2, 9), and (4, 12), the mean difference table can be constructed as follows

TABLE III: MEAN DIFFERENCE TABLE FOR EXAMPLE 3

x_k	$f(x_k)$	First-order	Second-order
0	0		
2	9	9/2	
4	12	3/2	-3/4

From the mean difference table we get

$$(y-3)^2 = \frac{9}{2}x - \frac{3}{4}x(x-2)$$

Thus by transforming it into the standard form we get the elliptic equation

$$\frac{(x-4)^2}{16} + \frac{(x-3)^2}{12} = 1 \quad \#$$

In high school analytic geometry, the point-slope equation of a line is the elementary form of Newton's interpolation polynomial. From the two examples above, we can use the same method to find the equations of parabola, ellipse, and hyperbola. Thus, it is a good way to apply Newton's interpolation polynomial to find equations of lines and curves in secondary school mathematics.

C. Application of Newton's interpolation polynomial to the constant equation

If both the left and right ends of a certain constant are k th degree polynomials with respect to a certain parameter x , according to Theorem 1, just pick $k+1$ different values of x to substitute into the left and right ends and consider whether they are equal.

Example 4: Prove a constant equation

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-a)(x-c)}{(b-a)(b-c)} + \frac{c^2(x-a)(x-b)}{(c-b)(c-a)} = x^2$$

Proof: Both sides of the constant are quadratic polynomials about x . Noting the characteristics of the left polynomial, make $x=a$, substitute into the two ends of the constant, we get: left = right = a^2 , similarly $x=b, x=c$ substitute into the left and right ends, still get left = right, so the constant equation holds. #

There are many other examples in secondary mathematics similar to the one above. Some problems are difficult to prove if you use constant deformation, but after using the result of Theorem 1, it is possible to narrow down the range to be considered to one of n special values, thus avoiding complicated polynomial deformation.

D. The use of Newton's interpolation method in solving systems of linear equations

Example 5: Solve the following system of equations for x :



$$\begin{cases} \frac{x}{a^3} - \frac{y}{a^2} + \frac{z}{a} = 1 \\ \frac{x}{b^3} - \frac{y}{b^2} + \frac{z}{b} = 1 \\ \frac{x}{c^3} - \frac{y}{c^2} + \frac{z}{c} = 1 \end{cases}$$

Solution: We deform the system of equations to obtain the following system of equations.

$$\begin{cases} x - ay + a^2z = a^3 \\ x - by + b^2z = b^3 \\ x - cy + c^2z = c^3 \end{cases}$$

We construct the polynomial

$$P_3(l) = x - yl + zl^2$$

$P_3(l)$ passes through the points $(a, a^3), (b, b^3), (c, c^3)$.

According to the following table of mean differences

TABLE IV: MEAN DIFFERENCE TABLE FOR EXAMPLE 5

x_k	$f(x_k)$	First-order	Second-order
a	a^3		
b	b^3	$a^2 + ab + b^2$	
c	c^3	$b^2 + bc + c^2$	$a + b + c$

we get the following equation.

$$\begin{aligned} P_3(l) &= a^3 - (a^2 + ab + b^2)(l - a) + (a + b + c)(l - a)(l - b) \\ &= (a + b + c)l^3 - (ab + bc + ac)l^2 + abc \end{aligned}$$

Comparing it with $P_3(l)$ we get

$$x = abc, y = ab + ac + bc, z = a + b + c.$$

For this type of linear equations, it is troublesome to use the simple linear equation system solution method, and even the matrix method of higher algebra is quite tedious to solve, while using the Newton interpolation method, not only is the operation speed fast, and not easy to make mistakes.

III. CONCLUSION

We discuss the application of Newton's interpolation

method in secondary school mathematics from four aspects respectively. Through examples and comparisons, we find that the use of Newton interpolation method does bring convenience to the solution of certain problems in secondary school mathematics. From the definition of mean difference to the table of mean differences, and even some theorems derived from it, it simplifies to a certain extent the problems of considerable difficulty and complexity, and provides a new way to solve them.

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