



Some Applications of Taylor's Formula in Numerical Analysis

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Abstract—The essence of Taylor's formula is a polynomial approach to approximate some complex functions. It is an approximation to a complex function. Taylor's formula has a wide range of applications in numerical analysis. In this paper we mainly discuss these applications of Taylor's formula in numerical analysis.

Index Terms—Numerical Analysis, Taylor's formula.

I. INTRODUCTION

Taylor's formula is named after the great British mathematician Brook Taylor. It first appeared in Taylor's work "The Incremental Method of Positive and Negative", published in 1715. Initially, Taylor did not give the expression for the remainder term of Taylor's formula and did not consider the convergence of Taylor's series. These problems were later satisfactorily solved by the intensive research of Peyano, Lagrange and Corsi. Taylor's formula is the key point and difficult point of higher mathematics, and has very important theoretical value. It has important applications in approximation calculations, limit calculations, inequality proofs, functional properties analysis, determinant calculations and grade convergence determination [1].

Numerical analysis is the study of numerical methods for solving various mathematical problems by computer. It focuses on how to perform approximate calculations using computer programming. In approximate calculations, simple functions are often used to approximate complex functions and keep their errors within allowable limits. And the essence of Taylor's formula is to approximate the function by a polynomial near the expansion point. Because of this feature of Taylor's formula, it has a wide range of applications in numerical analysis calculations [2-5]. This paper mainly summarizes some application problems of Taylor's formula in how to avoid error hazards, numerical integration and other problems.

II. SOME APPLICATIONS OF TAYLOR'S FORMULA IN NUMERICAL ANALYSIS

First, the common Taylor formula is given. Suppose the function $f(x)$ has $n+1$ order derivatives in the interval (a, b) , $x_0 \in (a, b)$, then for any $x \in (a, b)$, we have the following equation.

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

where the remaining items

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

here ξ is between x and x_0 .

Denote $f(x) = P_n(x) + R_n(x)$. In fact, Taylor polynomials can also be understood as interpolating polynomials that satisfy the interpolation condition with derivatives.

$$P_n^{(k)}(x_0) = f^{(k)}(x_0) (k=0, 1, \dots, n)$$

A. Using Taylor's formula to avoid error hazards

Numerically unstable algorithms are usually not used in numerical calculations. The algorithm should be designed to avoid the hazards of errors as much as possible. For example, when subtracting two similar numbers, Taylor's formula can be used to change the calculation formula, thus avoiding or reducing the loss of valid numbers.

Example 1: Let $x \approx y$. Calculate the value of $\ln x - \ln y$.

Analysis: If $\ln x$ and $\ln y$ are subtracted directly when $x \approx y$, there will be a loss of valid numbers. And if we set $f(x) = \ln x$, the number of conditions is

$$C_p = \left| \frac{xf'(x)}{f(x)} \right| = \left| \frac{1}{\ln x} \right|.$$

When $x \approx 1$, C_p is sufficiently large so that pathological problems are likely to occur. Therefore, if

$$\ln x - \ln y = \ln \frac{x}{y}$$

is used for the calculation, a pathological problem will occur because of $\frac{x}{y} \approx 1$. So we can use Taylor's formula to expand

$\ln x$ at y .

$$\ln x = \ln y + \frac{1}{y}(x-y) - \frac{1}{2y^2}(x-y)^2 + L$$

Thus we get

$$\ln x - \ln y = \frac{1}{y}(x-y) - \frac{1}{2y^2}(x-y)^2 + L \approx \frac{4xy - x^2 - 3y^2}{2y^2}.$$



B. Prove the trapezoidal formula using Taylor's formula

Theorem 1 [1] (Newton-Leibniz formula) If the function $F(x)$ is an original function of the continuous function $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

The Newton-Leibniz formula is also known as the fundamental formula of calculus. It is the main formula for calculating definite integrals. In the following, we use Taylor's formula to prove the trapezoidal formula.

$$\int_a^b f(x)dx \approx \frac{b-a}{2} [f(a) + f(b)]$$

Proof: Suppose $F(x)$ is an original function of the product function $f(x)$ in the interval $[a, b]$. We expand $F(x)$ at the points $x=a, x=b$ Taylor respectively.

$$\begin{aligned} F(x) &= F(a) + F'(a)(x-a) + \frac{F''(\xi_1)}{2!}(x-a)^2 \\ &= F(a) + f(a)(x-a) + \frac{f'(\xi_1)}{2!}(x-a)^2 \quad (1) \end{aligned}$$

$$\begin{aligned} F(x) &= F(b) + F'(b)(x-b) + \frac{F''(\xi_2)}{2!}(x-b)^2 \\ &= F(b) + f(b)(x-b) + \frac{f'(\xi_2)}{2!}(x-b)^2 \quad (2) \end{aligned}$$

Substituting $x = b$ into equation (1) yields

$$F(b) = F(a) + f(a)(b-a) + \frac{f'(\xi_1)}{2!}(b-a)^2. \quad (3)$$

Similarly, substituting $x = a$ into (2) gives

$$F(a) = F(b) + f(b)(a-b) + \frac{f'(\xi_2)}{2!}(a-b)^2. \quad (4)$$

(3)-(4) yields

$$F(b) - F(a) = \frac{b-a}{2} [f(a) + f(b)] - \frac{f''(\xi_3)}{12}(b-a)^3.$$

Therefore

$$\int_a^b f(x)dx = F(b) - F(a) \approx \frac{b-a}{2} [f(a) + f(b)]$$

Here the error is $-\frac{f''(\xi_3)}{12}(b-a)^3$, $\xi_3 \in (a, b)$.

C. Use Taylor's formula to find the approximation of the integral

In the calculation of the value of the definite integral, some of the original functions cannot be expressed in terms of primary functions or some of the original functions are very complicated to find or calculate, so the Newton-Leibniz formula cannot be used directly. Theoretically the definite integral is an objectively determined value, and the problem to be solved is whether other ways can be found to solve the approximate calculation of the definite integral. Taylor's formula is a great tool. It enables the approximate calculation of definite integrals.

Example 2: Find the approximate value of $\int_0^1 e^{-x^2} dx$ (to the nearest 10^{-5})

Solution: Since the product function of the definite integral e^{-x^2} is not available, the Taylor formula can be used to find its approximate value. Because

$$e^x = 1 + x + \frac{x^2}{2} + L + \frac{x^n}{n!} + L$$

therefore

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2} + L + \frac{(-1)^n x^{2n}}{n!} + L$$

$$\begin{aligned} \int_0^1 e^{-x^2} dx &= \int_0^1 1 dx - \int_0^1 x^2 dx + \int_0^1 \frac{x^4}{2} dx + L + \frac{(-1)^n}{n!} \int_0^1 x^{2n} dx + L \\ &= 1 - \frac{1}{3} + \frac{1}{2} \times \frac{1}{5} + L + \frac{(-1)^n}{n!} \times \frac{1}{(2n+1)} + L \end{aligned}$$

The above equation is an interleaved series. From the remaining terms, we know that $n=7$. Therefore

$$\int_0^1 e^{-x^2} dx \approx 0.746836$$

Example 3: Find the approximate value of $\int_0^1 \frac{\sin x}{x} dx$.

Solution: Since a is not available, it is expanded into a power series using Taylor's formula and then integrated one at a time, and then the integral value is calculated using the integrated series. Because

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\sin(\theta x + \frac{7\pi}{2})x^7}{7!} \\ \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{\sin(\theta x + \frac{7\pi}{2})x^6}{7!} \end{aligned}$$

so

$$\begin{aligned} \int_0^1 \frac{\sin x}{x} dx &= \int_0^1 1 dx - \int_0^1 \frac{x^2}{3!} dx + \int_0^1 \frac{x^4}{5!} dx - \int_0^1 \frac{\sin(\theta x + \frac{7\pi}{2})x^6}{7!} dx \\ &\approx 1 - \frac{1}{3 \times 3!} + \frac{1}{5 \times 5!} \approx 0.9416 \end{aligned}$$

Because of $\left| \sin(\theta x + \frac{7\pi}{2}) \right| \leq 1$, the error

$$|R_n(x)| \leq \frac{1}{7 \times 7!} < 0.5 \times 10^{-4}.$$

It is worth noting that because Taylor's formula is a local property, x cannot be too far from x_0 when using it for approximation calculations.

III. CONCLUSION

In this paper we have given the application of Taylor's formula in numerical problems such as error hazard avoidance and numerical integration. Taylor's formula can be used not only for constructing numerical algorithms, but also for analyzing the errors of numerical algorithms, which has a wide range of applications in numerical analysis.



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