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Several Methods of Summation of Number Series

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***Abstract*—In this paper, we introduce several methods of summation of number series by examples: Gauss method, term splitting method, term by term integration method, term by term derivation method, use of power series, Euler's constant method, use of Fourier series, etc.**

***Index Terms*—Number series, Summation, Fourier series.**

I. Introduction

In the nineteenth century, based on a detailed study of calculus, the French scientist A.L. Cauchy proposed the theory of series, which was further improved by N.H. Abel and K.T.H. Weierstrass by studying the problems. In today's society, the number series plays a role that cannot be ignored, and it is an essential tool for the study of analytical mathematics. It is an essential tool for the study of analytical mathematics, and the summation problem must be mentioned after the study of number series [1-3]. So, how to sum the number term series becomes an urgent problem to be solved. We will discuss the convergence and summation of number series using Gaussian method, definition method, term splitting method, term-by-term integration method, and Fourier series method.

II. Using Gaussian Method to Find the Sum of Series

Definition 1[1]: Given a series, the expression that joins its terms in order with a "+" sign, $u_1+u_2+\cdots+u_n+\ldots$, is called a constant term infinite series or a series of terms (also often abbreviated as a series). The number series is said to converge if the partial sum series $\left\{S_n=\sum_{i=1}^{n}u_i\right\}$ of the number series converges to S (i.e. $\lim_{n\to\infty}S_n=S$), and S is said to be the sum of the number series (1), denoted as $S=u_1+u_2+\ldots+u_n+\ldots$。 If $\{S_n\}$ is a divergent series, then the number term series is said to be divergent.

A. *Using Gaussian method to find the sum of series*

The so-called Gaussian method is to write out the desired series in reverse order, add it to the original series term by term, and then find the sum.

Example 1: Find the sum of 2 + 3 + 4 + ... + (n+1).

Solution: Let S= 2 + 3 + 4 + ... + (n+1), then again we get S=(n+1)+n +(n-1)+...+2, adding the two sides of the equation separately, we get $2S$ =(n+3)+(n+3)+(n+3)+...+(n+3)=n(n+3), so

$$S=\frac{n(n+3)}{2}.$$

B. *Using definition method to find the sum of series*

The essence of the definition method is to find the limit of the sum of the first n terms of a series.

Example 2: Find the sum of

$$S=\sum_{n=1}^{\infty}\frac{1}{n(n+2)}.$$

Solution: Let

$$a_n=\frac{1}{n(n+2)}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right).$$

Then

$$S_n=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+2}\right)=\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

This leads to

$$\lim_{n\to\infty}S_n=\frac{3}{4},\ S=\frac{3}{4}.$$

C. *Use the term splitting method to find the sum of the series*

We can split the term of the desired series into the sum of several series terms. Currently, some of the known common levels are

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\ln 2,\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12},\ \sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Example 3: Find the sum of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2(n+1)^2}.$$

Solution: Because

$$\frac{(-1)^{n+1}}{n^2(n+1)^2}=\frac{(-1)^{n+1}}{n^2}+\frac{(-1)^{n+1}}{(n+1)^2}-\frac{2(-1)^{n+1}}{n(n+1)},$$

we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2(n+1)^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)^2}-\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n(n+1)}$$

$$=\frac{\pi^2}{12}+\left(\frac{\pi^2}{12}-1\right)-2$$

$$=\frac{\pi^2}{6}-3.$$

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D. Use the term-by-term integration method to find the sum of the series

By the principle of invariant radius of convergence of series term by term integration, the original series is integrated term by term and then reduced to some easy to find power series, and then back to the derivative to find the sum of power series.

Example 4: Find the sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$.

Solution: We suppose

$$F(x)=\sum_{n=1}^{\infty} nx^{n-1}=1+2x+3x^2+\cdots+nx^{n-1}+\cdots$$

Next, we integrate $F(x)$ term by term over its convergence domain to obtain

$$\int_0^x F(t)dt=x+x^2+x^3+\cdots=\frac{x}{1-x},|x|<1.$$

So, we can get

$$F(x)=\sum_{n=1}^{\infty} nx^{n-1}=\left(\frac{x}{1-x}\right)'=\frac{1}{(1-x)^2}.$$

E. Use the term-by-term derivative method to find the sum of the positive series

If the power series $\sum_{n=1}^{\infty} a_n(x)$ converges in the interval (a, b) and $\sum_{n=1}^{\infty} a_n'(x)$ is continuous in (a, b). Then the series $\sum_{n=1}^{\infty} a_n(x)$ in (a, b) can be derived term by term, with

$$\left[\sum_{n=1}^{\infty} a_n(x)\right]'=\sum_{n=1}^{\infty} a_n'(x).$$

Example 5: Find the sum of the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$.

Solution: Let

$$S(x)=\sum_{n=1}^{\infty}\frac{x^n}{n}=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}+\cdots.$$

Its convergence interval is (-1, 1). So

$$s'(x)=1+x+x^2+\cdots+x^{n-1}+\cdots=\frac{1}{1-x}$$

Thus we get the result

$$s(x)=\int_0^x s'(t)dt=\int_0^x\frac{1}{1-t}dt=-\ln(1-x).$$

F. Use Taylor series method to find the sum of series

Definition 2[1]: Given $f\in C^{\infty}(a,b), x_0\in(a,b)$, Then the series $\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\cdots$ is said to be the Taylor series of $f(x)$ at point x_0, and its coefficients are called Taylor coefficients.

Example 6: Find the sum of the series $\sum_{n=1}^{\infty}\frac{5^n}{n!}$.

Solution: Let $f(x)=e^x$, expanding it to a Taylor series, and we obtain $e^x=\sum_{n=1}^{\infty}\frac{x^n}{n!}$. Thus we get

$$\sum_{n=1}^{\infty}\frac{5^n}{n!}=\sum_{n=1}^{\infty}\frac{x^n}{n!}\Bigg|_{x=5}=e^x\big|_{x=5}=e^5.$$

G. Using Euler's constant method to find the sum of series

The value of the formation limit $\lim_{n\to\infty}(\sum_{k=1}^{n}\frac{1}{k}-\ln n)$ is the Euler constant and is set to $c(c=0.57721\cdots)$.Then there are

$$\sum_{k=1}^{n}\frac{1}{k}=\ln n+c+a_n.$$

where $\lim_{n\to\infty} a_n=0$. Using this equation, we can find the sum of certain number series.

Example 7: Find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{n(2n+1)}$.

Solution: Let $S=\sum_{n=1}^{\infty}\frac{1}{n(2n+1)}$.

$$\lim_{n\to\infty} s=2-2\ln 2$$

$$=\sum_{k=1}^{n}(\frac{1}{k}-\frac{2}{2k+1})$$

$$=\sum_{k=1}^{n}\frac{1}{k}-2(\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2n+1})$$

$$=\sum_{k=1}^{n}\frac{1}{k}-2(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2n})-\frac{2}{2n+1}+2(1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2n})$$

$$=2\sum_{k=1}^{n}\frac{1}{k}-2\sum_{k=1}^{2n}\frac{1}{k}-\frac{2}{2n+1}+2$$

$$=2(c+\ln n+a_n)-2(c+\ln 2n+a_{2n})-\frac{2}{2n+1}+2$$

$$=2-2\ln 2+2a_n-2a_{2n}-\frac{2}{2n+1}$$

So $\lim_{n\to\infty} s=2-2\ln 2$, hence $s=2-2\ln 2$.

H. Using the Fourier series method to find the sum of several terms of the series

Definition 3[1]: If $f(x)$ has period 2π and is differentiable on the interval $[-\pi,\pi]$, then

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$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, n = 0,1,2,\cdots ,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, n = 1,2,\cdots .$$

We call a_n, b_n the Fourier coefficients of $f(x)$ and call

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

the Fourier series of $f(x)$. If $f(x)$ has a period of $2l$ and is differentiable over the interval $[-l, l]$, then the Fourier series can be obtained by replacing the variables $\frac{\pi x}{l} = t$ or $x = \frac{lt}{\pi}$ as the following.

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l} dx, n = 0,1,2,\cdots ,$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l} dx, n = 1,2,\cdots .$$

Theorem 1[1]:If the function of $f(x)$ with period 2π is smooth on $[-\pi, \pi]$ by segments, then at each point , the Fourier series of $f(x)$ converges to the arithmetic mean of the left and right limits of $\tilde{f}$ at the point x, i.e.

$$\frac{\tilde{f}(x+0) + \tilde{f}(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Here a_n, b_n is the Fourier series of $f(x)$. $\tilde{f}$ is a function of the extension of $f(x)$ to $(-\infty, +\infty)$ with period 2π .

Corollary 1[1]: If $f(x)$ is a continuous function with period 2π and smooth by segments on $[-\pi, \pi]$, then the Fourier series of $f(x)$ converges to $f(x)$ on $(-\infty, +\infty)$.

Example 8[4]: Find the sum of the series $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.

Solution: Let $f(x) = x^2$, and expand $f(x)$into a Fourier series over $[-\pi, \pi]$to obtain the following.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{4}{n^2}(-1)^n .$$

Therefore $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$.

Let $x = \pi$, we get

$$f(x) : \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx)$$

Since the Fourier series of $f(x)$ converges on $[-\pi, \pi]$ to $f(x)$, so

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = 0$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, so we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{(2n)^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2},$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = (1-\frac{1}{4})\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{3}{4}\cdot\frac{\pi^2}{6} = \frac{\pi^2}{8}.$$

III. Conclusion

For beginners, it is difficult to use Fourier series to find the sum of several terms because it is difficult to think which function's Fourier expansion can be used. However, this kind of problem is usually to find the Fourier series of a function first, and then use the Fourier series to find the sum of the terms, so it is only necessary to specify the Fourier series of the function at which point. In short, to find the sum of several terms is a key and a difficult problem, in addition to the methods discussed above, there are other methods such as differential equation method.

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