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On the Valuation of Integrals and the Study of Integral Inequalities

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***Abstract*—Integral valuation, as the name implies, is a simple estimation of the integral, estimating the upper and lower bounds of the integral. The purpose of this thesis is to apply several common integral inequalities to the valuation of definite integrals, including general integral valuation inequalities, convex function inequalities, Schwarz inequality, etc., and to present examples to deepen the understanding of integral valuation.**

***Index Terms*—Integral valuation, Schwarz inequality, Convex function inequalities.**

I. INTRODUCTION

Integral valuation, i.e., no detailed integration operations are performed, and only some simple techniques are used to obtain the approximate results of integration, which can be divided into two types: narrow and broad integral valuation. Narrow integral valuation, i.e., finding the upper limit or lower limit of the integral by simple calculation. The generalized integral valuation, on the other hand, does not require a detailed specific upper or lower bound result, but simply proves that the integral is greater than or less than (greater than or less than) another integral. Generalized integral valuation is widely used in real life, so we need to study some techniques to perform integral valuation faster and better [1-4].

II. DEFINITE INTEGRAL VALUATION INEQUALITIES

On $[a, b]$, if $f(x)$ is continuous, then we have $n \le f(x) \le N$ (n, N are the minimum and maximum values of $f(x)$ on the interval), and we have

$$n(b-a) \le \int_a^b f(x)dx \le N(b-a),$$

or

$$n \le \frac{\int_a^b f(x)dx}{b-a} \le N.$$

On $[a, b]$, if $f(x)$ and $g(x)$ are continuous and

$$ng(x) \le f(x)g(x) \le Ng(x),$$

(n, N are the minimum and maximum values of $f(x)$ on the interval),then

$$n\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le N\int_a^b g(x)dx$$

$$n \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le N.$$

Example 1： Estimate the approximate range of

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x^2 \sin x dx.$$

Solution: Because $x^2 \sin x$ is an increasing function on $[\frac{\pi}{4}, \frac{\pi}{2}]$, and the function $f(x) = x^2 \sin x$ has extreme values on it.

$$f_{\max}(x) = f(\frac{\pi}{2}) = (\frac{\pi}{2})^2 \sin\frac{\pi}{2} = \frac{\pi^2}{4}$$

$$f_{\min}(x) = f(\frac{\pi}{4}) = (\frac{\pi}{4})^2 \sin\frac{\pi}{4} = \frac{\sqrt{2}\pi^2}{32}$$

Therefore, the upper limit of the definite integral is

$$M = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\pi^2}{4} dx = \frac{\pi^2}{4} \times \frac{\pi}{4} = \frac{\pi^3}{16}.$$

And the lower limit is

$$m = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sqrt{2}\pi^2}{32} = \frac{\sqrt{2}\pi^2}{32} \times \frac{\pi}{4} = \frac{\sqrt{2}\pi^3}{128}.$$

Thus we get

$$\frac{\sqrt{2}\pi^3}{128} < \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x^2 \sin x dx < \frac{\pi^3}{16}.$$

Example 2: Prove the inequality

$$\frac{2}{5} \le \int_{2\pi}^{\frac{5\pi}{2}} \frac{2\sin x}{x} dx \le \pi.$$

Proof: On the interval $\left(2\pi, \frac{5}{2}\pi\right)$, since $f(x) = \frac{2\sin x}{x}$ is constantly greater than zero, and

$$f'(x) = \frac{2\cos x(x - \tan x)}{x^2}.$$

So $f(x)$ is monotonically decreasing in the interval, so we have

$$f\left(\frac{5\pi}{2}\right) < \frac{4}{5\pi} < f(x) = \frac{2\sin x}{x} < f(2\pi) < 2,$$

$$\frac{2}{5} = \int_{2\pi}^{\frac{5\pi}{2}} \frac{4}{5\pi} dx < \int_{2\pi}^{\frac{5\pi}{2}} \frac{\sin x}{x} dx < \int_{2\pi}^{\frac{5\pi}{2}} 2dx = \pi.$$

Example 3: Let $f(x)$ be a non-negative monotone non-increasing continuous function on $[0,1]$ (i.e.,

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$f(x) > f(y)$ holds if $x < y$). Show that if $0 < \alpha < \beta < 1$, the following inequality holds.

$$\int_0^{\alpha} f(x)dx \geq \frac{\alpha}{\beta}\int_{\alpha}^{\beta} f(x)dx.$$

Proof: From the question set and the median theorem, we know that

$$\int_{\alpha}^{\beta} f(x)dx = f(\xi_1)(\beta-\alpha) \leq f(a)(\beta-a),(\alpha \leq \xi_1 \leq \beta).$$

Thus we get

$$\frac{1}{\alpha}\int_0^{\alpha} f(x)dx \geq f(a) > \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x)dx.$$

Therefore,

$$\frac{1}{\alpha}\int_0^{\alpha} f(x)dx \geq f(a) > \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(x)dx$$

can be obtained.
And since $0 < \alpha < \beta < 1$, so

$$1-\frac{\alpha}{\beta} < 1.$$

So we get

$$\int_0^{\alpha} f(x)dx \geq \frac{\alpha}{\beta}\int_{\alpha}^{\beta} f(x)dx.$$

As we can see from the above example, the value estimated by the definite integral valuation inequality is extremely wrong with the actual maximum and minimum values of the function. The definite integral valuation inequality can be used for any simple valuation of definite integrals.

III. VALUATION OF CONVEX FUNCTION INEQUALITIES

Definition 1[1]. A function $f(x)$ is said to be a lower convex function on the interval I if it is defined on the interval I and the inequality

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

holds for every points $x_1, x_2 \in I, x_1 \neq x_2$ and every $\lambda \in (0,1)$. If the function $-f(x)$ is a lower convex function on the interval I, then $f(x)$ is an upper convex function on the interval. The general term for the upper convex function and the lower convex function is called the convex function.

Property 1.Let $f(x)$ be a convex function on the interval I. Then for any three points $x_1 < x_2 < x_3$ on I, the inequality

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

holds.

Property 2.Let $f(x)$ be a convex function on the interval , then $f'(x)$ is an increasing function on I and for any two points x_1,x_2 on I , the inequality

$$f(x_2) \geq f(x_1) + f'(x_1)(x_2-x_1)$$

holds.

Property 3. Let $f(x)$ be a convex function on the interval. Then there are two finite one-sided derivatives of $f(x)$ everywhere on the interval I, and the inequality $f_-'(x) \leq f_+'(x)$ holds.

Property 4. Let $f(x)$ be a second-order derivable function on an open interval I, then the sufficient condition for $f(x)$ to be a convex function on an open interval I is $f''(x) \geq 0, x \in I$.

Example 4: Prove the following inequality.

$$\int_0^1 1+xdx \leq \int_0^1 2^x dx \leq \int_0^1 1+x^3 dx$$

Proof: Let $f(x) = 1+x^3-2^x, g(x) = 2^x-1-x.$ Then $f''(x) = 6x-2^x(\ln 2)^2 \geq 0, g''(x) = 2^x(\ln 2)^2 > 0.$,

So $f(x)$ and $g(x)$ are both convex functions on $[0,1]$. $f'(x)$and $g'(x)$ are increasing functions on I, so that $f(x)$ and $g(x)$ have minimum values on the interval which are taken at $x = 0$.
Because of

$$f(0) = 1+0^3-2^0 = 0, g(0) = 2^0-1-0 = 0$$
$$f(x) \geq f(0) = 0, g(x) \geq g(0) = 0$$

we get

$$\int_0^1 1+x^3-2^x dx \geq \int_0^1 f(0)dx = 0,$$

i.e.

$$\int_0^1 (1+x^3-2^x)dx \geq 0$$
$$\int_0^1 1+x^3dx \geq \int_0^1 2^x dx.$$

The same reasoning leads to

$$\int_0^1 2^x dx \geq \int_0^1 1+xdx\cdot$$

Example 5: Prove that

$$\int_0^1 \sin \pi xdx \leq \int_0^1 \frac{\pi^2}{2}x(1+x)dx.$$

Proof: Let

$$f(x) = \sin \pi x, g(x) = \frac{\pi^2}{2}x(1+x),$$
$$F(x) = \frac{\pi^2}{2}x(1+x) - \sin\pi x.$$

It is known that

$$f(x) \geq 0, g(x) \geq 0,$$
$$F'(x) = \pi^2 x + \frac{\pi^2}{2} - \pi\cos\pi x,$$
$$F''(x) = \pi^2 + \pi^2 \sin \pi x \geq 0 (x \in [0,1]),$$

Therefore,

$$F(x) = \frac{\pi^2}{2}x(1+x) - \sin \pi x$$

is a convex function, and the rest of the cases are similar to Example 4.

IV. SCHWARZ INEQUALITY

Lemma 1[1]: If $f(x)$ and $g(x)$ are differentiable on $[a, b]$, then the inequality

$$[\int_a^b f(x)g(x)dx]^2 \leq \int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx$$

holds.

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Example 6: Prove that if $f(x)$ is differentiable and $f(x) \geq n \geq 0$ holds ($x \in [a,b]$) then you have inequality

$$\int_a^b f(x)dx \cdot \int_a^b \frac{1}{f(x)}dx \geq (b-a)^2 \cdot$$

Proof: By the fact that $f(x)$ is differentiable and $f(x) \geq n \geq 0$ holds, so $\frac{1}{f(x)}, \sqrt{f(x)}, \frac{1}{\sqrt{f(x)}}$ are differentiable.

According to Schwarz's inequality, there is

$$\int_a^b f(x)dx \cdot \int_a^b \frac{1}{f(x)}dx \geq (\int_a^b \sqrt{f(x)} \cdot \frac{1}{\sqrt{f(x)}}dx)^2 = (\int_a^b dx)^2 = (b-a)^2$$

holds.

Example 7: If $f(x)$ is productable on $[a, b]$, then

$$\left(\int_a^b f(x)dx\right)^2 \leq (b-a)\int_a^b f^2(x)dx \cdot$$

Proof: By Schwarz inequality we get

$$[\int_a^b f(x)dx]^2 = [\int_a^b f(x) \cdot 1dx]^2 \leq \int_a^b 1^2 dx \cdot \int_a^b f^2(x)dx = (b-a)\int_a^b f^2(x)dx.$$

V. CONCLUSION

We use several common and very important integral inequalities to perform integral valuation operations, including definite integral valuation inequalities, convex function inequalities, Schwarz inequalities, etc. Using integral inequalities for integral valuation, rather than directly computing the integral, not only shortens the integral valuation procedure, but also saves a lot of time and even improves the accuracy of the integral valuation.

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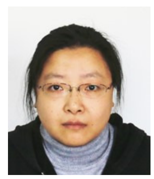
Authors' biography with Photos



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