ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 8, Issue 12, December 2022

Research on the Application of Taylor Formula

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***Abstract*—Taylor formula is a very important part of higher mathematics. Its basic idea is to use polynomials to approximate known functions, and the coefficients of this polynomial are determined by the derivatives of each order of a given function. Its theoretical method has become an important tool for studying function limits and estimating errors. Taylor formula embodies the essence of function "approximation method", which can simplify complex problems. Taylor formula's function of "simplifying complexity" has played an important role in many aspects of mathematics.**

This paper discusses the relevant definitions of Taylor's publicity, and discusses in detail the practical application of Taylor's formula in finding approximate values and estimation errors, finding limits, proving the median problem, distinguishing extreme values, etc. The application of Taylor formula in many aspects can improve our understanding of the formula and play an important role in opening up ideas for solving problems.

***Index Terms*—Taylor formula, Advanced mathematics, Mean value theorem, Limit, Higher derivative.**

I. INTRODUCTION

Taylor formula is an extremely important tool to study the problems related to calculus, so its research (including theoretical proof and application promotion) has a lasting appeal to mathematical workers. Taylor formula not only gives polynomial approximation of function, but also gives the expression of residual term (i.e. error). With the completeness, preciseness of its theoretical system and operability in specific applications, Taylor formula provides a powerful tool for future generations of mathematicians to study problems related to integration [1-5].

This paper first introduces the definition of Taylor formula and its proof method; Then, the application of Taylor's formula in higher mathematics is explored through examples, including using Taylor's formula to find the approximate value and estimation error, finding the limit, proving the median problem, judging the extreme value of the function, judging the concavity and convexity of the function and the inflection point, proving the inequality, etc.

II. SEVERAL FORMS OF TAYLOR FORMULA

2.1 Taylor Formula with Peano Type Remainder

Let $f(x)$ have a derivative of order n at point x_0,

that is, there is $f^{(n)}(x_0)$, then there is a neighborhood $U(x_0)$, and for any point x in the neighborhood, there is

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \Lambda + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o[(x - x_0)^n].$$

We call $R_n(x) = o[(x-x_0)^n]$ a Peano type remainder, and the above formula is called Taylor formula with Peano type remainder. Polynomial

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \Lambda + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called Taylor polynomial of $f(x)$ at x_0.

2.2 Taylor formula with Lagrange type remainder

If the function $f(x)$ has a derivative up to the $n+1$ order in a neighborhood $U(x_0,\delta)$, then for any $x \in U(x_0,\delta)$, there is at least a point ξ between x_0 and x, so that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \Lambda + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1},$$

where $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$ is called the Lagrange type remainder of $f(x)$. The above formula is

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

IJARMATE

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 8, Issue 12, December 2022

called Taylor formula with Lagrange type remainder at point x_0.

2.3 Other forms of remainder and Maclaurin formula

(1) Cauchy remainder

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{n!}(x-\xi)^n(x-x_0)$$

where ξ is between x and x_0.

(2) Integral remainder

$$R_n(x)=\frac{\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n\,dt}{n!}.$$

Special form of Taylor formula:

When $x_0=0$, Taylor's formula is

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}$$

$$+\Lambda+f^{(n)}(0)\frac{x^n}{n!}+R_n(x).$$

$R_n(x)$ is the corresponding remainder, which is called the Maclaurin expansion of Taylor formula, also called the Maclaurin formula; Maclaurin formula is mainly used in some special functions, such as trigonometric function, logarithmic function, etc.

Some commonly used Taylor expansions can be obtained by using Maclaurin expansion, such as

$$e^x=1+x+\frac{x^2}{2!}+\Lambda+\frac{x^n}{n!}+\frac{e^{\theta x}}{(n+1)!}x^{n+1}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\Lambda+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+2})$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\mathrm{L}+(-1)^n\frac{x^{2n}}{(2n)!}+o(x^{2n})$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\Lambda+(-1)^n\frac{x^{n+1}}{n+1}+o(x^{n+1})$$

$$\frac{1}{1-x}=1+x+x^2+\Lambda+x^n+o(x^n).$$

III. Application Examples of Taylor Formula

Taylor formula has been widely used in many fields. This paper mainly introduces the application of Taylor formula in higher mathematics through examples.

3.1 Use Taylor formula to find the approximate value and estimate error

According to the remainder of Taylor expansion, the error caused when Taylor formula is used to approximate a function can be estimated concretely.

The Lagrange type remainder is

$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}.$$

if $\left|f^{(n+1)}(x)\right|\le M$ and M are a certain number, the rest will not exceed $\frac{M}{(n+1)!}\left|x-x_0\right|^{n+1}$. From this, we can approximately calculate some numerical values and estimate their errors.

Example 1. Calculate the approximate value of $\sqrt[3]{30}$ and estimate its error.

Solution. Obviously, $\sqrt[3]{27+3}=3\sqrt[3]{1+\frac{1}{9}}$.

Let $f'(x)=\sqrt[3]{1+x}$, then $f'(x)=\frac{1}{3}(1+x)^{-\frac{2}{3}}$,

$$f''(x)=-\frac{2}{9}(1+x)^{-\frac{5}{3}},\ f'''(x)=\frac{10}{27}(1+x)^{-\frac{8}{3}},$$

$$f^{(4)}(x)=-\frac{80}{81}(1+x)^{-\frac{11}{3}}.$$

So we have

$$f(0)=1,\ f'(0)=\frac{1}{3},$$

$$f''(0)=-\frac{2}{9},\ f'''(0)=\frac{10}{27}.$$

It can be seen from Formula (4) that

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+x^3\frac{f^{(4)}(\theta x)}{3!}x^4$$

$$=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3+\frac{-\frac{80}{81}}{4!}(1+\theta x)^{-\frac{11}{3}}x^4$$

where $0<\theta<x$.

Thus we obtain

$$\sqrt[3]{1+\frac{1}{9}}=1+\frac{1}{3}\times\frac{1}{9}-\frac{1}{9}\times\frac{1}{81}+\frac{5}{81}\times\frac{1}{729}$$

$$+\frac{-80}{81\times 4!}\left(1+\frac{1}{9}\theta\right)^{-\frac{11}{3}}\left(\frac{1}{9}\right)^4.$$

Therefore we have

$$\sqrt[3]{30}=\sqrt[3]{27+3}=3\sqrt[3]{1+\frac{1}{9}}$$

$$=3\left(1+\frac{1}{3}\times\frac{1}{9}-\frac{1}{9}\times\frac{1}{81}+\frac{5}{81}\times\frac{1}{729}\right)+\frac{-240}{81\times 4!}\left(1+\frac{1}{9}\theta\right)^{-\frac{11}{3}}\left(\frac{1}{9}\right)^4$$

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 8, Issue 12, December 2022

$$\approx 3\left(1+\frac{1}{3}\times\frac{1}{9}-\frac{1}{9}\times\frac{1}{81}+\frac{5}{81}\times\frac{1}{729}\right)\approx 3.107252.$$

The error is

$$R=\left|\frac{-240}{81\times 4!}\left(1+\frac{1}{9}\theta\right)^{-\frac{11}{3}}\left(\frac{1}{9}\right)^{4}\right|$$

$$\leq\frac{240}{81\times 24}\times\frac{1}{9^4}\approx 1.88\times 10^{-5}.$$

3.2 Use Taylor formula to find the limit

For the undetermined limit problems, the L·Hospital rule can generally be used to solve them. However, for some cases where derivation is complicated and calculation is complex, especially when the L·Hospital rule is used for many times, it is much easier to use Taylor's formula to calculate the limit. The limit is calculated by Taylor formula, which is generally in the form of Maclaurin formula and Taylor formula with Peano type remainder. When the limit formula is a fraction, it is generally required that the numerator and denominator develop into Maclaurin formula of the same order, and the limit can be obtained by comparison.

Example 2. Calculate the following limits

$$\lim_{x\to o}\frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x-e^{x^2})\sin^2 x}.$$

Solution. If L·Hospital's law is adopted to solve this problem, it will be very troublesome, so the following solution is adopted: known by Taylor's formula

$$\sqrt{1+x^2}=(1+x^2)^{\frac{1}{2}}=1+\frac{1}{2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4+o(x^4)$$

$$\cos x=1-\frac{x^2}{2!}+o(x^2)$$

$$e^{x^2}=1+x^2+o(x^2)$$

Because when $x\to 0$, $\sin x\approx x$, so we have

$$\lim_{x\to o}\frac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x-e^{x^2})\sin^2 x}$$

$$=\frac{\frac{x^2}{2}+1-\left[1+\frac{1}{2}x^2-\frac{1}{8}x^4+o(x^4)\right]}{\left[1-\frac{x^2}{2!}-1-x^2+o(x^2)\right]x^2}$$

$$=\lim_{x\to 0}\frac{\frac{1}{8}x^4+o(x^4)}{-\frac{3}{2}x^4+o(x^4)}=-\frac{1}{12}.$$

3.3 Use Taylor formula to prove the median problem

Example 3. Let the function $f(x)$ have a third order continuous derivative on $[-1,1]$, and $f(-1)=0$, $f(1)=1$, $f'(0)=0$. Prove that there is at least one point ξ in $(-1,1)$, so that $f'''(\xi)=3$.

Prove. According to Maclaurin formula, we have

$$f(x)=f(0)+f(0)'x+\frac{1}{2!}f''(0)x^2+\frac{f'''(\eta)}{3!}x^3,$$

where η is between 0 and x.

Let $x=-1$ and $x=1$ respectively, and subtract the two obtained equations to get

$$f'''(\eta_1)+f'''(\eta_2)=6,\ (-1<\eta_1<0,\ 0<\eta_2<1)$$

From the continuity of $f'''(x)$, there is a maximum value M and a minimum value m on $[\eta_1,\eta_2]$, so

$$m\leq\frac{1}{2}[f'''(\eta_1)+f'''(\eta_2)]\leq M.$$

It is known from the intermediate value theorem of continuous functions that there is at least one point $\xi\in[\eta_1,\eta_2]\subset(-1,1)$, so that

$$f'''(\xi)=\frac{1}{2}[f'''(\eta_1)+f'''(\eta_2)]=3.$$

3.4 Taylor formula in proving the existence of roots

Example 4. Let $f(x)$ be second order differentiable on $[a,+\infty)$, and $f(a)>0, f'(a)<0$. For $x\in(a,+\infty), f''\leq 0$. Prove that the equation $f(x)=0$ has a unique real root in $(a,+\infty)$.

Proof. Because $f''(x)\leq 0$, $f'(x)$ decreases monotonically, and $f'(a)<0$, so when $x>a$, we nave $f'(x)<f'(a)<0$, so $f(x)$ decreases monotonically on $(a,+\infty)$ strictly. At point a, expand it with Taylor's formula, and we get

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(\xi)}{2}(x-a)^2(a<\xi<x).$$

Let $f'(a)<0, f'(\xi)\leq 0$ by the question, then $\lim\limits_{x\to\infty}=-\infty$, so there must be $b>a$, so that $f(b)<0$. And because $f(a)>0$, applying the intermediate value theorem of continuous functions on $[a,b]$, there is

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 8, Issue 12, December 2022

$x_0 \in (a,b)$, so that $f(x_0)=0$, and x_0 is unique from the strict monotonicity of $f(x)$, so the equation $f(x)=0$ has a unique real root in $(a,+\infty)$. [6] examined the development and refinement of possible mathematical models for the intellectual system of career guidance. Mathematical modeling of knowledge expression in the career guidance system, Combined method of eliminating uncertainties, Chris-Naylor method in the expert information system of career guidance, Shortliff and Buchanan model in the expert information system of career guidance and DempsterSchafer in the expert information system of career guidance method has been studied. [7] discussed that according to the observations in this paper, an existing mathematical model of banking capital dynamics should be tweaked. First-order ordinary differential equations with a "predator-pray" structure make up the model, and the indicators are competitive.

3.5 Judge extreme value with Taylor formula

Example 5. Let $f(x)$ be first order differentiable in the neighborhood $U(x_0;\delta)$, second order differentiable at $x=x_0$, and $f'(x_0)=0$, $f''(x_0)\neq 0$.

(i) If $f''(x_0)<0$, then $f(x)$ gets the maximum value at x_0.

(ii) If $f''(x_0)>0$, then $f(x)$ takes a minimum at x_0.

Proof. Second order Taylor formula of $f(x)$ at x_0 can be obtained from conditions

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+o((x-x_0)^2)$$

Since $f'(x_0)=0$, so we have

$$f(x)-f(x_0)=[\frac{f''(x_0)}{2}+o(1)](x-x_0)^2. \quad (*)$$

Because $f''(x_0)\neq 0$, there is a positive number $\delta' \leq \delta$, when $x\in U(x_0;\delta')$, $\frac{1}{2}f''(x_0)$ and $\frac{1}{2}f''(x_0)+o(1)$ have the same sign. Therefore, when $f''(x_0)<0$, the above equation (*) takes a negative value, so that for any $x\in U(x_0;\delta')$, we have $f(x)-f(x_0)<0$, that is, $f(x)$ gets a maximum value at x_0. Similarly, for $f''(x_0)>0$, it can be obtained that $f(x)$ gets a minimum value at x_0.

3.6 Judge concavity and convexity of function and inflection point by Taylor formula

Taylor formula can also be used to study concavity and convexity of functions and inflexion points, in which the following lemma is required.

Lemma 1. Let $f(x)$ be continuous on $[a,b]$ and have first and second derivatives on (a,b). If $f''(x)>0$ for $x\in(a,b)$, then the graph of $f(x)$ on $[a,b]$ is convex.

Lemma 2. If $f(x)$ is differentiable in some inner n-order and satisfies

$$f'(x_0)=f''(x_0)=\Lambda=f^{(n-1)}(x_0)=0,$$
$$f^{(n)}(x_0)\neq 0,(n>2)$$

Then we have the following results.

1) If n is odd, then $(x_0,f(x_0))$ is the inflection point;

2) If n is an even number, then $(x_0,f(x_0))$ is not an inflection point.

Example 6. Determine whether $(0,4)$ is the inflection point of $f(x)=e^x+e^{-x}+2\cos x$?

Solution. $f'(x)=e^x-e^{-x}-2\sin x$, $f'(0)=0$.

$f''(x)=e^x+e^{-x}-2\cos x$, $f''(0)=0$.

$f'''(x)=e^x-e^{-x}+2\sin x$, $f'''(x)=0$.

$f^{(4)}(x)=e^x+e^{-x}+2\cos x$, $f^{(4)}(0)=4\neq 0$.

Since $n=4$ is an even number, $(0,4)$ is not the inflection point of $f(x)$.

IV. Conclusion

Taylor formula is a very important content in mathematical analysis and an indispensable tool for studying various fields of mathematics. This article is a preliminary arrangement on the basis of consulting a large number of materials about Taylor's formula. This article mainly makes a systematic introduction to Taylor's formula in terms of approximate value calculation, limit calculation, extreme value of discriminant function, concavity and convexity of discriminant function and inflection point, which reflects the important position of Taylor's formula in the application of differential calculus. Through the discussion of the above aspects, we can make full use of its problem-solving skills to achieve twice the result with half the effort.

It is worth mentioning that although Taylor's formula has been applied to many mathematical fields, there are also many aspects that are rarely mentioned and need to be explored constantly. Taylor formula is also a very practical tool in the practical application of mathematics. Only by mastering these knowledge, and constantly training and summarizing on this basis, can we skillfully apply Taylor

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 8, Issue 12, December 2022

formula, find ways to solve problems from different angles, and explore new methods to solve problems, so that we can better and more conveniently study some complex functions and solve more practical mathematical problems.

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