

# A New Variant of the Classical Newton's Method with Cubic Convergence

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**Abstract**—In this paper, we present a new third-order iterative method which is a variant of the classical Newton's method. It requires two evaluations of the functions and one evaluation of the derivative per iteration. The efficiency index of the presented method is  $\sqrt[3]{3}$ , which is better than that of the classical Newton's method  $\sqrt{2}$ . Several numerical examples are given to show the efficiency.

**Key words**—Newton's method, cubic convergence, nonlinear equations.

## I. INTRODUCTION

We are concerned with iterative methods to find a simple root  $\alpha$  of a nonlinear equation  $f(x) = 0$ , where  $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  for an open interval  $I$  is a scalar function. The classical Newton's method is one of the well-known and widely used iterative methods given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (1)$$

which converges quadratically in some neighborhood of  $\alpha$ .

In recent years, many iterative methods have been developed for solving nonlinear equation and especially many modifications of the classical Newton's method that are free from second derivatives have been proposed and analyzed, see [2-17] and the references therein. Many numerical applications use high precision in their computation, so higher-order numerical methods are important [18]. In this paper, we first present a new variant of the classical Newton's method with cubic convergence. Then, numerical examples are given to show the performance. [4] discussed that In surgical planning and cancer treatment, it is crucial to segment and measure a liver tumor's volume accurately. Because it would involve automation, standardisation, and the incorporation of complete volumetric information, accurate automatic liver tumor segmentation would substantially affect the processes for therapy planning and follow-up reporting. Based on the Hidden Markov random field, Automatic liver tumor detection in CT scans is possible using hidden Markov random fields (HMRf-EM). [7] discussed that Liver tumor division in restorative pictures has been generally considered as of late, of which the Level set models show an uncommon potential with the advantage of overall optima and functional effectiveness. [13] discussed that Tumor segmentation required also the identical automatic initialization as

regarding the liver. This phase was applied only in order to liver volume, obtained following automatic delineation of lean meats surface: this latter, used to original dataset quantity, was used as a new mask in order to be able to prevent processing overloads and even avoid errors related to be able to arsenic intoxication surrounding tissues delivering similar gray scale droit.

## II. CONVERGENCE ANALYSIS

Now, we consider the iteration scheme

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \frac{2f(\frac{x_n+y_n}{2})f(x_n)}{f'^2(x_n)} \end{cases} \quad (2)$$

which is a variant of the classical Newton's method.

**Theorem** Let  $\alpha$  be a simple zero of sufficiently differentiable function  $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  for an open interval  $I$ . If  $x_0$  is sufficiently close to  $\alpha$ , then the method defined by (2) is of third-order and satisfies the error equation

$$e_{n+1} = (4c_2^2 - \frac{7}{4}c_3)e_n^3 + O(e_n^4),$$

where  $e_n = x_n - \alpha$  and  $c_k = f^{(k)}(\alpha) / k! f'(\alpha)$ .

**Proof** Using Taylor expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)], \quad (3)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + O(e_n^3)]. \quad (4)$$

Furthermore, we can obtain

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \quad (5)$$

$$f(\frac{x_n+y_n}{2}) = f'(\alpha)[\frac{1}{2}e_n + \frac{3}{4}c_2e_n^2 - \frac{1}{2}(c_2^2 - \frac{9}{4}c_3)e_n^3 + O(e_n^4)] \quad (6)$$

and

$$2f'(\frac{x_n+y_n}{2}) = f'(\alpha)[1 + 3c_2e_n - 3(c_2^2 - \frac{9}{4}c_3)e_n^2 + O(e_n^3)]. \quad (7)$$

From (4,5,7), we obtain

$$e_{n+1} = (4c_2^2 - \frac{7}{4}c_3)e_n^3 + O(e_n^4).$$

This means the method defined by (2) is of third-order. That completes the proof.

Now, we consider the efficiency index defined as  $p^{1/\omega}$ , where  $p$  is the order of the iterative method and  $\omega$  is the number of function evaluations per iteration needed by the method. It is easy to see that the efficiency index of the new method defined by (2) is  $\sqrt[3]{3}$  which is better than that of the classical Newton's method  $\sqrt{2}$ .

### III. NUMERICAL EXAMPLES

In this section, we employ the new method defined by (2) to solve some nonlinear equations and compare them with the classical Newton's method (NM). Displayed in Table I are the number of iterations (IT) and the number of function evaluations (NFE) required such that  $|f(x_n)| < 10^{-15}$ .

We use the following functions:

$$f_1(x) = (\sin x)^2 - x^2 + 1,$$

$$\alpha = 1.40449164821534111524670,$$

$$f_2(x) = (x+2)e^x - 1,$$

$$\alpha = -0.442854401002388542440968,$$

$$f_3(x) = e^x \sin x + \ln(x^2 + 1),$$

$$\alpha = 0.$$

The computational results presented in Table 1 show that, the presented methods converge more rapidly than Newton's method and require the less NFE. Therefore, the new methods (2) have better convergence efficiency.

Table I: Comparison of various iterative methods

	$x_0$	IT(NM) )	NFE(NM)	IT(Eq.(2) )	NFE(Eq.(2) )
$f_1(x)$	0.5	10	20	4	12
	1.2	5	10	3	9
	2	6	12	4	12
$f_2(x)$	0	6	12	4	12
	0.5	5	10	3	9
	-0.5	7	14	4	12
$f_3(x)$	-1	8	16	3	9
	0.5	7	14	4	12

2 7 14 4 12

### REFERENCES

- [1] A. M. Ostrowski, Solution of Equations in Euclidean and Banach Space, Academic Press, New York, 1973.
- [2] F.A. Potra, V. Ptk, Nondiscrete induction and iterative processes, Research Notes in Mathematics, vol. 103, Pitman, Boston, 1984.
- [3] S. Weerakoon and T.G.I. Fernando, A variant of Newtons method with accelerated third-order convergence, Appl. Math. Lett. 13 (2000)87-93.
- [4] Christo Ananth, S. Amutha, K. Niha, Djabbarov Botirjon Begimovich, "Enhancing Segmentation Approaches from Super Pixel Division Algorithm to Hidden Markov Random Fields with Expectation Maximization (HMR-EM)", International Journal of Early Childhood Special Education, Volume 14, Issue 05, 2022, pp. 2400-2410.
- [5] A.Y. O'zban, Some new variants of Newtons method, Appl. Math. Lett. 17 (2004) 677-682.
- [6] C. Chun, Some third-order families of iterative methods for solving nonlinear equations, Appl. Math. Comput. 188 (2007) 924-933.
- [7] Christo Ananth, M Kameswari, Densy John Vadakkan, Dr. Niha.K., "Enhancing Segmentation Approaches from Fuzzy-MPSO Based Liver Tumor Segmentation to Gaussian Mixture Model and Expected Maximization", Journal Of Algebraic Statistics, Volume 13, Issue 2, June 2022, pp. 788-797.
- [8] J. Kou, Y. Li, X. Wang, A modification of Newton method with third-order convergence, Appl. Math. Comput. 181 (2006) 1106-1111.
- [9] J. Kou, Y. Li, X. Wang, Third-order modification of Newtons method, J. Comput. Appl. Math. 205 (2007) 1-5.
- [10] J. Kou: The improvements of modified Newton's method, Appl. Math. Comput. 189 (2007), p. 602-609
- [11] C. Chun: A simply constructed third-order modifications of Newton's method, J. Comput. Appl. Math. 219(2008), p. 81-89.
- [12] C. Chun: Some second-derivative-free variants of super-Halley method with fourth-order convergence, Appl. Math. Comput. 192 (2007), p. 537-541.
- [13] Christo Ananth, Bindhya Thomas, Priyanka Surendran, Dr.A.Anitha, "Enhancing Segmentation Approaches from GC-GGC to Fuzzy K-C-Means", Annals of the Romanian Society for Cell Biology, Volume 25, Issue 4, April 2021, pp. 2829 - 2834.
- [14] J.F.Traub, Iterative Methods for the Solution of Equations, Chelsea publishing company, New York, 1977.
- [15] YoonMee Ham, Changbum Chun, A fifth-order iterative method for solving nonlinear equations, Appl. Math. Comput. 194 (2007) 287-290.
- [16] J. Kou, Y. Li, X. Wang, Some modifications of Newtons method with fifth-order convergence, J. Comput. Appl. Math. 209 (2007) 146-152.
- [17] E Cárdenas, Castro R, Sierra W. A Newton-type midpoint method with high efficiency index[J]. Journal of Mathematical Analysis and Applications, 2020, 491(2):124381.
- [18] M. Grau, J.L. Díaz-Barrero, An improvement to Ostrowski root-finding method, Appl. Math. Comput. 173 (2006) 450-456.