ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol.7, Issue 6, June 2021

Research on the Solution of Inverse Matrix of Matrix Polynomial

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***Abstract*—Matrix theory is one of the important contents of linear algebra, and it is also an important tool to deal with practical problems. Many practical problems can be solved with the idea of matrix, which is simple and fast. Although many linear algebra materials involve matrix polynomials, there is no concrete method to find the inverse matrix of matrix polynomials. In this paper, we study the theory of the greatest common factor of polynomials and the related knowledge of matrices, and give several methods to solve the inverse matrix of general matrix polynomials.**

***Index Terms*—Matrix polynomial, inverse matrix, greatest common factor, matrix, undetermined coefficient method.**

I. INTRODUCTION

Matrix polynomials are abstract, widely applicable, and generally instructive and abstract. It has an inherent and inevitable relationship with elementary mathematics. The object of higher algebra course is algebra, and one form of studying algebra is polynomial, which guides students' development from concrete to abstract from the aspect of mathematical thought [1-5]. Polynomial is the necessary knowledge and theory of higher algebra, and is an important content in higher algebra. A comprehensive understanding of polynomial theory is the basic requirement of higher algebra. The theory of polynomials has an unusual guiding role in the learning of elementary algebra. With the profound reform of teaching reform, this guiding role has become more and more prominent, showing the strong vitality of polynomial theory. Matrix polynomial theory can help students understand more mathematical ideas and methods, cultivate strong interest in learning mathematics, and improve students' mathematical thinking ability. In this paper, combined with the theory of the greatest common factor of polynomials and the related knowledge of matrices, the inverse matrix of matrix polynomials is given.

II. THE SOLUTION OF INVERSE MATRIX OF MATRIX POLYNOMIAL

A. *Definition of matrix polynomial*

Definition 1 Let

$$f(x)=a_0x^n+a_1x^{n-1}+\Lambda\ +a_{n-1}x+a_n$$

be a polynomial of degree n with respect to the unknown quantity x, A is a square matrix, and E is a unit matrix of the same order of A, then

$$f(A)=a_0A^n+a_1A^{n-1}+\Lambda\ +a_{n-1}A+a_nE$$

is called the matrix polynomial of A formed by the polynomial

$$f(x)=a_0x^n+a_1x^{n-1}+\Lambda\ +a_{n-1}x+a_n,$$

which is denoted as $f(A)$.

For example, if

$$f(x)=x^3-3x^2+2x+5,$$

$$A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},$$

then

$$f(A)=A^3-3A^2+2A+5E$$
$$=\begin{pmatrix}5 & -1\\ 0 & 5\end{pmatrix}$$

is a polynomial of matrix A. Obviously, matrix polynomials are also matrices.

B. *The definition of inverse matrix of matrix polynomial*

Definition 2 Let A be a square matrix of order n over the number field P, and $f(A)$ be a polynomial of matrix A. If there is a matrix polynomial $g(A)\in P[x]$ such that

$$f(A)g(A)=g(A)f(A)=E,$$

then the matrix polynomial $f(A)$ is invertible, and the matrix polynomial $g(A)$ is the inverse of the matrix polynomial $f(A)$.

C. *Solutions of inverse matrix of matrix polynomial*

Definition 3 Let A be an n-order square matrix over the number field P, $f(x)\in p[x]$, and $p[x]$ be a polynomial ring of one variable over the number field P, then $f(A)$ is called the polynomial of matrix A.

Definition 4 Let A be an n-order square matrix over the number field P, denote $A=(a_{ij})_{n\times n}$, then

ISSN (ONLINE): 2454-9762
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Available online at www.ijarmate.com

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$$f(\lambda)=|\lambda E-A|$$

$$=\begin{vmatrix}\lambda-a_{11} & -a_{12} & \Lambda & -a_{1n}\\ -a_{21} & \lambda-a_{22} & \Lambda & -a_{2n}\\ \Lambda & \Lambda & \Lambda & \Lambda\\ -a_{n1} & -a_{n2} & \Lambda & \lambda-a_{nn}\end{vmatrix}$$

$$=\lambda^n-(a_{11}+a_{22}+\Lambda\ a_{nn})\lambda^{n-1}+\Lambda+(-1)^n|A|$$

is called characteristic polynomial of matrix A.

The inverse matrix of matrix polynomials is widely used in linear algebra, and the solution of inverse matrix is also different. Next, we introduce several methods to solve the inverse matrix of matrix polynomial.

In order to calculate the inverse matrix of some matrix polynomials which are easy to simplify or simple in form, we can use undetermined coefficient method or decomposition factor method to find its inverse.

(1) Decomposition factor method

Example 1 If A, B are two square matrices of order n and $AB-2A-3B=-5E$. Prove that $A-3E$ is invertible, and find the inverse matrix of $A-3E$.

Proof. Since $AB-2A-3B=-5E$, so

$$(A-3E)B-2A+6E=E,$$

thus

$$(A-3E)(B-2E)=E.$$

Therefore $A-3E$ is reversible, and

$$(A-3E)^{-1}=B-2E.$$

(2) Undetermined coefficient method

Example 2 If matrix A satisfies $A^3=3E$, matrix $B=A^2-2A+3E$. Prove that B is invertible and find its inverse matrix.

Proof. Since the highest degree of the inverse matrix of B can only be quadratic, so we can let

$$B^{-1}=aA^2+bA+cE.$$

It can be seen from the condition that

$$E=BB^{-1}=aA^4+(-2a+b)A^3$$
$$+(3a-2b+c)A^2+(3b+2c)A+3cE.$$

It follows from $A^3=3E$ that

$$(3a-2b+c)A^2+(3a+3b-2c)A+(-6a+3b+3c)E$$
$$=E.$$

So we have

$$\begin{cases}3a-2b+c=0,\\ 3a+3b-2c=0,\\ -6a+3b+3c=1.\end{cases}$$

Solve the equation system to get

$$a=\frac{1}{66}, b=\frac{3}{22}, c=\frac{5}{22},$$

therefore

$$B^{-1}=\frac{1}{66}(A^2+9A+15E).$$

(3) The solution of inverse matrix of general matrix polynomial

Although the decomposition factor method and undetermined coefficient method for solving the inverse matrix of matrix polynomial are relatively simple, they need some operation skills, and they are not applicable to all inverse matrices of matrix polynomials. For example: let matrix

$$A=\begin{pmatrix}4 & 6 & 0\\ -3 & -5 & 0\\ -9 & -11 & 1\end{pmatrix},$$

$$g(A)=A^3+2A^2-5A-6E,$$

find the inverse matrix of $g(A)$. This problem can not be solved by decomposition factor method, and although it can be solved by undetermined coefficient method, the calculation is complicated. In the following, we give a common method to find the inverse of a general matrix polynomial.

There are the following theorems about the invertibility of the polynomial of a matrix and the solution of the inverse matrix

Theorem 1. If A is an n-order square matrix, C is a complex field, $f(x), g(x) \in C[x]$, $f(A)=0$, and the roots of equation $f(x)=0$ are eigenvalues of matrix A of order n, then the necessary and sufficient condition for $g(A)$ to be invertible is $(f(x), g(x))=1$. There exists $u(x), v(x) \in C[x]$, such that

$$u(x)f(x)+v(x)g(x)=1,$$

and $[g(A)]^{-1}=v(A)$.

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Available online at www.ijarmate.com

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Proof. Necessity. If $\lambda_1, \lambda_2, \Lambda, \lambda_n$ are all eigenvalues of A, then $f(\lambda_1), f(\lambda_2), \Lambda, f(\lambda_n)$ are all eigenvalues of $f(A)$, and $g(\lambda_1), g(\lambda_2), \Lambda, g(\lambda_n)$ are all eigenvalues of $g(A)$. Since $g(A)$ is reversible, so $|g(A)| = \prod_{i=1}^{n} g(\lambda_i) \neq 0$, but since $f(A) = 0$, then $f(\lambda_i) = 0 \ (i = 1,2,\Lambda, n)$, $f(x)$ and $g(x)$ have no common zero point, namely $(f(x), g(x)) = 1$.

Sufficiency: If $(f(x), g(x)) = 1$, then $f(A) = 0$, for each λ_i, there must be $f(\lambda_i) = 0$ and $g(\lambda_i) \neq 0 \ (i = 1,2,\Lambda, n)$, i.e., $|g(A)| = \prod_{i=1}^{n} g(\lambda_i) \neq 0$, so $g(A)$ is reversible. When $f(x)$ and $g(x)$ are mutually prime, there must be $u(x), v(x) \in C[x]$, such that $u(x)f(x) + v(x)g(x) = 1$, $v(A)g(A) = E$, thus

$$v(A) = [g(A)]^{-1}.$$

Example 3. We know that matrix A satisfies the formula $A^3 = 3E$ and matrix $B = A^2 - 2A + 3E$. Prove that B is invertible, and find its inverse matrix.

Solution. Let $f(x) = x^3 - 3, g(x) = x^2 - 2x + 3,$ then $f(A) = 0$, and the roots of $f(x)$ are the eigenvalue of A, and because $f(x)$ and $g(x)$ have no common root, the two polynomials $f(x)$ and $g(x)$ are mutually prime, that is, $(f(x), g(x)) = 1$. According to theorem 2, $g(A)$ is reversible, and

$$-\frac{1}{66}(x+7)f(x) + \frac{1}{66}(x^2 + 9x + 15)g(x) = 1$$

is obtained by the method of rolling and division, so

$$[g(A)]^{-1} = \frac{1}{66}(A^2 + 9A + 15E).$$

In particular, if $g(x)$ represents a polynomial of degree one, then if $f(x) = q(x)g(x) + r$, where r is a nonzero constant. Then $g(A)q(A) = -rE$, that is,

$$[g(A)]^{-1} = -\frac{1}{r}q(A).$$

Example 4. Let A be a square matrix. Prove that $A^2 + 3A - 4E = 0$ is true, and find $(A + 5E)^{-1}$.

Solution. If $f(x) = x^2 + 3x - 4, g(x) = x + 5,$ then $f(A) = 0$ can be obtained from the meaning of the title, and the roots of $f(x)$ are eigenvalues of A. By using the comprehensive division method we have

$$f(x) = (x-2)g(x) + 6.$$

Therefore, we have

$$[g(A)]^{-1} = (A + 5E)^{-1} = -\frac{1}{6}(A - 2E).$$

Corollary 1. If A is a square matrix of order n and $g(x) \in C[x]$. $f_A(x)$ is a characteristic polynomial of A, then $g(A)$ is invertible if and only if $(f_A(x), g(x)) = 1$. At this time, there is $u(x), v(x) \in C[x]$, so that $u(x)f(x) + v(x)g(x) = 1$, and $[g(A)]^{-1} = v(A)$.

Proof. Let $\lambda_1, \lambda_2, \Lambda, \lambda_n$ be all eigenvalues of A, then $g(\lambda_1), g(\lambda_2), \Lambda, g(\lambda_n)$ are all eigenvalues of $g(A)$, so $g(A)$ is invertible if and only if

$$|g(A)| = \prod_{i=1}^{n} g(\lambda_i) \neq 0,$$

that is, for every λ_i, , there is $g(\lambda_i) \neq 0 \ (i = 1,2,\Lambda, n)$. Since $f_A(\lambda_i) = 0$ for each λ_i, therefore $g(\lambda_i) \neq 0$, if and only if $(f_A(x), g(x)) = 1$.

Using the above conclusion, if $g(A)$ is invertible and $f(A) = 0$, the inverse matrix of $g(A)$ can be obtained by Theorem 1.

Example 5. Given the matrix

$$A = \begin{pmatrix} 4 & 6 & 0 \\ -3 & -5 & 0 \\ -9 & -11 & 1 \end{pmatrix},$$

$$g(A) = A^3 + 2A^2 - 5A - 6E.$$

Find the inverse matrix of $g(A)$.

Solution. Let $f_A(x) = (x-1)^2(x+2)$, $f_A(x)$ be the characteristic polynomial of A, and then set $g(x) = (x+1)(x-2)(x+3)$, it is obvious that $(f_A(x), g(x)) = 1$. Corollary 1 shows that $g(A)$ is reversible, and since

$$(-x^2 + x + 10)f_A(x) + (x^2 - 3x - 2)g(x) = 32,$$

we can get

$$[g(A)]^{-1} = \frac{1}{32}(A^2 - 3A - 2E).$$

Corollary 2. If A is a square matrix of order n and $g(x) \in C[x], m(x)$ is the minimum polynomial of A, then $g(A)$ is invertible if and only if $(m(x), g(x)) = 1$.

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

IJARMATE

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol.7, Issue 6, June 2021

At this point, there is $u(x), v(x) \in C[x]$, so that $u(x)m(x) + v(x)g(x) = 1$, and $[g(A)]^{-1} = v(A)$.

Corollary 3. If A is a square matrix of order n, $f(x), g(x) \in C[x], f(A) = O$, and $f(x) \mid f_A(x)$, where $f_A(x)$ is the characteristic polynomial of A, then $g(A)$ is invertible if and only if $(f(x), g(x)) = 1$. At this point, there is $u(x), v(x) \in C[x]$, so that $u(x)f(x) + v(x)g(x) = 1$, and $[g(A)]^{-1} = v(A)$.

Example 6. Let matrix

$$A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ 7 & 8 & a \end{pmatrix},$$

and

$$f(x) = x^2 + x - 2,$$

$$g(A) = A^3 + 2A^2 - 5A - 6E,$$

find the inverse matrix of $g(A)$.

Solusion. Let $f_A(x) = (x-1)(x+2)(x-a)$ be a characteristic polynomial of A, since B $f(x) \mid f_A(x)$, and $f(x)$ and $g(x) = (x+1)(x-2)(x+3)$ are mutually prime, from the inference 3 of theorem 2, $g(A)$ is invertible, and since

$$(-x^3 + 2x^2 + 9x - 10)f(x) + (x^2 - 3x - 2)g(x) = 32,$$

therefore

$$[g(A)]^{-1} = \frac{1}{32}(A^2 - 3A - 2E).$$

According to theorem 1, if two polynomials are not prime, then there are $u(x), v(x)$ so that $v(x)g(x) = d(x)$ if $f(x) = 0, g(x) \neq 0$. If x is replaced by matrix A, and $f(A) = 0, g(A) \neq 0$, then $v(A)g(A) = d(A)$. If $d(A)$ is reversible, then

$$\left[g(A)^{-1}\right] = [d(A)]^{-1} v(A).$$

Thus $[d(A)]^{-1}v(A)$ is the inverse of $g(A)$ matrix polynomial. Thus we can obtain the following theorem 2.

Theorem 2. If A is a square matrix of order n, C is the complex field, $f(x), g(x) \in C[x]$, and $f(A) = 0$, $g(x) \neq 0$, $(f(x), g(x)) = d(x)$, then there is $u(x), v(x) \in C[x]$ such that

$$u(x)f(x) + v(x)g(x) = d(x).$$

If $g(A)$ is invertible, $d(A)$ must be invertible; If $d(A)$ is invertible, $g(A)$ must be invertible. When $g(A)$ is reversible, we obtain $[g(A)]^{-1} = [d(A)]^{-1}v(A)$.

Proof. Suppose that $g(x) = d(x)g_1(x)$, then $g(A) = d(A)g_1(A)$. Obviously, if $g(A)$ is reversible, $d(A)$ must be reversible too. On the contrary, since $(f(x), g(x)) = d(x)$, there exists $u(x), v(x) \in C[x]$, such that

$$u(x)f(x) + v(x)g(x) = d(x).$$

Furthermore, since $f(A) = 0$, then $v(A)g(A) = d(A)$. If $d(A)$ is invertible, then $g(A)$ must be invertible, and when $g(A)$ is invertible, there is

$$[g(A)]^{-1} = [d(A)]^{-1}v(A).$$

The proof is completed.

In theorem 2, in order to find $g(A)^{-1}$, we need to find $[d(A)]^{-1}$. In specific problems, how should $[d(A)]^{-1}$ be solved? The following theorem can solve this problem.

Theorem 3. If A is a square matrix of order n, C is the complex field, and $f(x), g(x) \in C[x], f(A) = 0$. If $g(A)$ is invertible, then there exists $v_i(x) \in C[x]$ $(i = 1, 2, \Lambda, n)$, such that

$$[g(A)]^{-1} = \prod_{i=1}^{n} v_i(A).$$

Proof. If $(f(x), g(x)) = d_1(x) = 1$, then there are $u_1(x), v_1(x) \in C[x]$, such that

$$u_1(x)f(x) + v_1(x)g(x) = 1.$$

According to the title condition, $f(A) = 0$, then $[g(A)]^{-1} = v_1(A)$ can be obtained from theorem 1.

If $(f(x), g(x)) = d_1(x) \neq 1$, let

$$f(x) = d_1(x)f_1(x), \; g(x) = d_1(x)g_1(x),$$

then from the title condition, we can get $f(A) = 0$. So $g(A)$ is reversible.Therefore, $d_1(A), g_1(A)$ are reversible,and $f_1(A) = 0$.

Since $(f_1(x), g_1(x)) = 1$, so there is $v_1(x) \in C[x]$, such that $[g_1(A)]^{-1} = v_1(A)$.

Let $f(x) = d_1^{k_1}(x)f_{11}(x)$, where $d_1(x)$ cannot divide $f_{11}(x)$. By the title condition $f_{11}(A) = 0$, and $\partial(f(x)) > \partial(f_{11}(x))$.

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

IJARMATE
International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol.7, Issue 6, June 2021

If $(d_1(x), f_{11}(x)) = d_2(x) = 1$, then therre is $v_2(x) \in C[x]$, such that $[d_1(A)]^{-1} = v_2(A)$. Thus we have

$$[g(A)]^{-1} = [g_1(A)]^{-1}[d_1(A)]^{-1} = v_1(A)v_2(A).$$

If $(d_1(x), f_{11}(x)) = d_2(x) \neq 1$, let

$$f_{11}(x) = d_2(x)f_2(x), d_1(x) = d_2(x)g_2(x).$$

In this way, we can do it in turn, and because $\partial(f(x)) > \partial(f_{11}(x)) > \Lambda \geq 0$, when we do a certain step, there must be $(f_{n-1n-1}(x), d_{n-1}(x)) = 1$, and $[d_{n-1}(A)]^{-1} = v_n(A)$.

To sum up, it can be concluded that

$$[g(A)]^{-1} = [g_1(A)]^{-1}[g_2(A)]^{-1} \Lambda\, [g_{n-1}(A)]^{-1}[d_{n-1}(A)]^{-1}$$
$$= \prod_{i=1}^{n} v_i(A).$$

Example 7. Let A represent a square matrix of order n, polynomials $f(x) = x^5 + 3x^4 - 7x^3 - 15x^2 + 18x$, and $g(x) = x^3 + 5x^2 + 6x$, and $f(A) = 0$. When $g(A)$ is invertible, find the inverse of $g(A)$.

Solusion. First of all,

$$(f(x), g(x)) = x^2 + 3x = d_1(x)$$

can be calculated by the method of rolling phase division. Let

$$f(x) = d_1(x)f_1(x), g(x) = d_1(x)g_1(x),$$

then $f_1(x) = x^3 - 7x + 6$. And since $f(A) = 0$, we can get $f_1(A) = 0$.

Then, $\frac{1}{12}f_1(x) - \frac{1}{12}(x^2 - 2x - 3)g_1(x) = 1$, can be obtained by the method of rolling phase division. Therefore, we have

$$[g_1(A)]^{-1} = -\frac{1}{12}(A^2 - 2A - 3E).$$

Obviously $d_1(x)$ can't divide $f_1(x)$. And since

$$(f_1(x), d_1(x)) = x + 3 = d_2(x),$$

let

$$f_1(x) = d_2(x)f_2(x), d_1(x) = d_2(x)g_2(x),$$

then

$$f_2(x) = x^2 - 3x + 2,\ g_2(x) = x.$$

And since $f(A) = 0$, so we have $f_2(A) = 0$.

Then we can get $\frac{1}{2}f_2(x) - \frac{1}{2}(x-3)g_2(x) = 1$ by the method of rolling phase division. And then we have $[g_2(A)]^{-1} = \frac{1}{2}(3E - A)$. Obviously $d_2(x)$ can't divide $f_2(x)$. Because $(f_2(x), d_2(x)) = 1$ and

$$\frac{1}{20}f_2(x) - \frac{1}{20}(x-6)d_2(x) = 1,$$

we can get

$$[d_2(A)]^{-1} = \frac{1}{20}(6E - A).$$

From Theorem 3, we can get

$$[g(A)]^{-1} = [g_1(A)]^{-1}[g_2(A)]^{-1}[d_2(A)]^{-1}$$
$$= -\frac{1}{480}(A^4 - 11A^3 + 33A^2 - 9A - 54E).$$

III. Conclusions

In this paper, through the study of the theory of the greatest common factor of polynomials, several methods to solve the inverse matrix of matrix polynomial matrix are obtained. Although these methods are relatively easy to understand, but the knowledge involved contains many theories of polynomials and matrices, and the calculation skills are relatively strong. Understanding and mastering these methods can enhance the understanding of Higher Algebra related knowledge, broaden thinking, so as to better apply higher mathematics to practice.

Acknowledgment

This paper is supported by the 12th teaching reform and research project of Taishan University (Project No. 201914).

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ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol.7, Issue 6, June 2021

