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Orthogonal transformation and partial derivative methods for simplifying quadratic form into standard form

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***Abstract*—As a basic course of mathematics, higher Algebra plays an important role in the course system. quadratic form is one of the important contents in Higher Algebra. For the sake of convenience, when we use quadratic form to solve problems, we often need to look for a linear substitution to change it into the form of square sum, that is, to simplify quadratic form into standard form, it's also a difficult point. There is a close relation between the transformation of quadratic form into standard form and the diagonalization of Matrix, so the problem of transforming quadratic form into the similar diagonalization of symmetric Matrix is solved. On the basis of a brief introduction of quadratic form, this paper mainly introduces the orthogonal transformation method and the partial derivative method of transforming quadratic form into standard form, it makes dealing with quadratic problem more diversity and flexibility.**

***Index Terms*—Quadratic form, canonical form, linear substitution, orthogonal transformation, partial derivative.**

I. INTRODUCTION

Advanced Algebra is an important basic course for mathematics majors in university, and it plays an important role in the course system. And quadratic form is one of the important contents of Advanced Algebra, and its theory is widely used in mathematics, physics and so on. One of the key points of the quadratic form is to find a linear substitution to change it into the form of the sum of squares,i.e., the standard form of the quadratic form [1-7] .

In addition to the matching method and elementary transformation, we can use orthogonal linear substitution to transform any real quadratic form into a standard form. In addition, the quadratic form can be changed into the standard form by using the partial derivative method. Although the partial derivative method is similar to the formula method, it does not need to observe the formula, but is a method, the process of solution is also simpler than the methods of formulation, elementary transformation and orthogonal transformation, so it is a simple and feasible method.

In this paper, the typical examples to deepen the understanding of the process and methods of proof, and can increase the diversity and flexibility in dealing with the problem, should master the method given.

II. TRANSFORMING REAL QUADRATIC FORM INTO STANDARD FORM BY ORTHOGONAL TRANSFORMATION

A. *Orthogonal transformation method*

Theorem 1 For any real symmetric matrix A with order n, there exists an orthogonal matrix T with order n such that $T^{'}AT = T^{-1}AT$ is diagonal.

Since the matrices of the real quadratic form are real symmetric matrices, by Theorem 1, the solution of the orthogonal matrix can be carried out as follows:

Step 1. Let $\lambda_1, \lambda_2, \Lambda, \lambda_r$ be all the different eigenvalues of the matrix A of quadratic form.

Step 2. For every eigenvalue λ_i of A, the solution of homogeneous system of linear equations $(\lambda_i E - A)X = 0$, we get a basic solution system, that is, a set of bases of the characteristic subspace V_{λ_i} of A. Based on this group of bases, a set of orthonormal basis of V_{λ_i} is obtained by using the Gram-Schmidt process method.

Step 3. Since $\lambda_1, \Lambda, \lambda_r$ are different each other, the vector groups $\eta_{11}, \Lambda, \eta_{1k_1}, \Lambda\ \eta_{r1}, \Lambda, \eta_{rk_r}$ are still orthogonal to each other, so they form a group of orthonormal basis of R^n. Thus, the orthogonal matrix T can be obtained, and thus the quadratic form can be simplified to the standard form.

Example 1. Transform real quadratic form

$$f(x_1, x_2, x_3) = 2x_1x_2 + 2x_1x_3 - 2x_1x_4 - 2x_2x_3 + 2x_2x_4 + 2x_3x_4$$

into standard form.

Solution. The matrix of $f(x_1, x_2, x_3)$ is

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 0 \end{bmatrix}.$$

The corresponding characteristic equation is

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$$|\lambda E - A| = \begin{vmatrix} \lambda & -1 & -1 & 1 \\ -1 & \lambda & 1 & -1 \\ -1 & 1 & \lambda & -1 \\ 1 & -1 & -1 & \lambda \end{vmatrix}$$

$$= -(\lambda-1)^3 \begin{vmatrix} 1 & 1 & -1-\lambda \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (\lambda-1)^3(\lambda+3).$$

The characteristic root is 1(triple) , -3 .

For eigenvalue 1, we solve the equations $(E-A)X = 0$, i.e.,

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

to obtain a set of fundamental solutions

$$(1,1,0,0)', (1,0,1,0)', (-1,0,0,1)',$$

that is, a set of bases of the eigensubspace V_1 corresponding to eigenvalue 1. By orthogonalizing and uniting them, we get

$$\begin{cases} \alpha_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)', \\ \alpha_2 = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0)', \\ \alpha_3 = (-\frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{3}{\sqrt{12}})'. \end{cases}$$

For eigenvalue -3 , solve the equations $(-3E-A)X = 0$, that is,

$$\begin{pmatrix} -3 & -1 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ -1 & 1 & -3 & -1 \\ 1 & -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

and get a set of basic solution systems

$$\alpha_4 = (1, -1, -1, 1)',$$

that is, a set of bases of eigensubspace V_{-3} corresponding to eigenvalue -3, and unitize them to get

$$\alpha_4 = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})'.$$

The eigenvectors $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ constitute a set of orthonormal bases of R^4, so the orthonormal matrix is

$$T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{12}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{12}} & -\frac{1}{2} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{12}} & -\frac{1}{2} \\ 0 & 0 & \frac{3}{\sqrt{12}} & \frac{1}{2} \end{bmatrix}.$$

Therefore we have

$$T'AT = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -3 \end{bmatrix},$$

and $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^3 - 3x_4^2$.

B. *A variant of orthogonal transformation method*

The method of transforming real quadratic form into standard form by orthogonal transformation is given above. In this method, the eigenvalues corresponding to quadratic matrix need to be obtained first, and the objective of simplifying quadratic form can be achieved by solving linear equations for many times, which is cumbersome. This paper introduces an elementary transformation method which uses orthogonal transformation to transform quadratic form into standard form. This method does not need to solve characteristic equations and linear equations, which is simple and easy. Firstly, the related theory is given.

Theorem 2 The characteristic matrix $\lambda E - A$ of n order matrix A is equivalent to a diagonal matrix $B(\lambda)$, that is, there are invertible matrices $P(\lambda)$ and $Q(\lambda)$, such that $P(\lambda)(\lambda E - A)Q(\lambda) = B(\lambda)$, where

$$B(\lambda) = \begin{bmatrix} \varphi_1(\lambda) & & & \\ & \varphi_2(\lambda) & & \\ & & \ddots & \\ & & & \varphi_n(\lambda) \end{bmatrix}.$$

If $f(\lambda) = \lambda E - A$, there is obviously

$$f(\lambda) = \varphi_1(\lambda)\varphi_2(\lambda)\Lambda\ \varphi_n(\lambda).$$

Theorem 3 A square matrix A with order n is similar to a diagonal matrix if and only if the elementary factors of A are all linear.

Theorem 4 The real symmetric matrix A must be similar to the diagonal matrix.

Theorem 5 Let λ_1, λ_2 be the two eigenvalues of a real symmetric matrix A, and P_1, P_2 be the corresponding

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eigenvectors. If $\lambda_1 \neq \lambda_2$, then P_1 and P_2 are orthogonal. From theorem 2 and theorem 3, when A is similar to a diagonal matrix, all the factorizations of $\varphi_i(\lambda)$ $(i=1,2,\Lambda\ ,n)$ are first order factors.

Let λ_i be the γ multiple eigenvalue of the real n order symmetric matrix A, then $f(\lambda_i)=0$, and there are and only γ are zero in $\varphi_1(\lambda_i),\Lambda\ ,\varphi_n(\lambda_i)$. Let

$$\varphi_{i_1}(\lambda_i)=\Lambda\ =\varphi_{i_r}(\lambda_i)=0,$$

where $i_1,\Lambda\ ,i_r$ are some r number in $1,2,\Lambda\ ,n$, and the rest $\varphi_i(\lambda_i)\neq 0$, then

$$P(\lambda_i)(\lambda_i E-A)(q_{i_1}(\lambda_i)q_{i_2}(\lambda_i)\Lambda\ q_{i_r}(\lambda_i))=0_{n\times r}$$

can be obtained.

Similarly, the eigenvectors corresponding to other eigenvalues can be obtained. If A is a n order real symmetric matrix, it is known from Theorem 5 that the corresponding n eigenvectors obtained by the above method are two orthogonal unit vectors, and the matrix composed of these n vectors is the orthogonal matrix that transforms the quadratic form into the standard form.

In order to find $Q(\lambda)$, while the elementary transformation of $\lambda E-A$ into diagonal matrix $B(\lambda)$, the unit matrix E only does the Elementary matrix, then when $\lambda E-A$ into $B(\lambda)$, E becomes $Q(\lambda)$. That is,

$$\begin{bmatrix}\lambda E-A\\ E\end{bmatrix}\to\begin{bmatrix}B(\lambda)\\ Q(\lambda)\end{bmatrix}.$$

Example 2. Find an orthogonal transformation that transforms the quadratic

$$2x_1^2+5x_2^2+5x_3^2+4x_1x_2-4x_1x_3-8x_2x_3$$

into a standard form.

Solution. The matrix of quadratic form is

$$A=\begin{bmatrix}2 & 2 & -2\\ 2 & 5 & -4\\ -2 & -4 & 5\end{bmatrix}.$$

$$\begin{bmatrix}\lambda E-A\\ E\end{bmatrix}=\begin{bmatrix}\lambda-2 & -2 & 2\\ -2 & \lambda-5 & 4\\ 2 & 4 & \lambda-5\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\to$$

$$\begin{bmatrix}-2 & 0 & 0\\ 0 & \lambda-1 & 0\\ 0 & 0 & -\frac{1}{2}(\lambda-1)(\lambda-10)\\ 1 & \frac{1}{2}(\lambda-5) & -\frac{1}{2}(\lambda-9)\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

So

$$B(\lambda)=\begin{bmatrix}-2 & 0 & 0\\ 0 & \lambda-1 & 0\\ 0 & 0 & -\frac{1}{2}(\lambda-1)(\lambda-10)\end{bmatrix},$$

$$Q(\lambda)=\begin{bmatrix}1 & \frac{1}{2}(\lambda-5) & -\frac{1}{2}(\lambda-9)\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}.$$

Obviously the characteristic value of A is $\lambda_{1,2}=1$ (double), and $\lambda_3=10$.

Since $\phi_1(1)=\phi_2(1)=0$, when $\lambda_1=\lambda_2=1$, so

$$q^1(1)=\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}, q^2(1)=\begin{bmatrix}4\\ -1\\ -1\end{bmatrix}$$

is a pair of linearly independent eigenvectors of the eigenvalue 1, and we get

$$p_1=\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},$$

$$p_2=\frac{1}{3\sqrt{2}}\begin{bmatrix}4\\ -1\\ 1\end{bmatrix},$$

by applying the Gram-Schmidt process method to them.

When $\lambda_3=10$, $\phi_3(10)=0$, so the corresponding eigenvector is $q^3(10)=\begin{bmatrix}-\frac{1}{2}\\ -1\\ 1\end{bmatrix}$. Unifying it gives

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$$p_3=\begin{bmatrix}-\frac{3}{4}\\-\frac{3}{2}\\\frac{3}{2}\end{bmatrix}.$$

Let $P=(p_1,\ p_2,\ p_3)$, then P is the matrix of orthogonal transformation which converting quadratic form into standard form, so we get the orthogonal transformation $X=PY$, such that

$$f(x_1,x_2,x_3)=y_1^2+y_2^2+10y_3^2.$$

III. Partial Derivative Method

The orthogonal transformation method and the orthogonal transformation method of transformation from real quadratic form to standard form are introduced. But the operation process of orthogonal transformation method is complicated. In order to master the knowledge of quadratic form quickly and flexibly, develop mathematical thinking and cultivate the ability of multi-angle thinking, the partial derivative method of turning quadratic form into standard form will be given through examples.

Let the quadratic form $f(x_1,x_2,\ldots,x_n)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}x_ix_j$.

There are three scenarios for discussion.

(1) If at least one of a_{ij} $(i=1,2,\ldots,n)$ is not zero, such as $a_{11}\neq 0$, then $f(x_1,x_2,\ldots,x_n)=\frac{1}{2}\frac{\partial f}{\partial x_1}$ is calculated. Let $f=\frac{1}{a_{11}}f_1^2+g$, where g no longer contains x_1, then let

$$\begin{cases}y_1=f_1\\y_2=x_2\\\Lambda\\y_n=x_n.\end{cases}.$$

And then find out $g_1=\frac{1}{2}\frac{\partial g}{\partial x_2}$. Let $g=\frac{1}{a_{22}}g_1^2+h$, where x_2 does not exist in h, and a_{22} is the coefficient of x_2 in g_1, so $f=\frac{1}{a_{11}}f_1^2+\frac{1}{a_{22}}g_1^2+h$. Make

$$\begin{cases}z_1=g_1\\z_2=y_2\\\Lambda\\z_n=y_n.\end{cases}$$

Following this sequence, the quadratic form can be transformed into a standard form.

(2) If all $a_{ii}=0\,(i=1,2,\Lambda,n)$, but at least one $a_{ij}\neq 0\,(j>1)$, without loss of generality, let $a_{12}\neq 0$, compute

$$f_1=\tfrac{1}{2}\frac{\partial f}{\partial x_1},\ f_2=\tfrac{1}{2}\frac{\partial f}{\partial x_2},$$

and let

$$f=\frac{1}{a_{12}}\left[(f_1+f_2)^2-(f_1-f_2)^2\right]+\phi,$$

where φ does not contain x_1,x_2. At this point, make

$$\begin{cases}y_1=f_1+f_2,\\y_2=f_1-f_2,\\y_3=x_3,\\\Lambda\ \Lambda\ \Lambda\\y_n=x_n.\end{cases}$$

Observe the structure of ϕ, if ϕ still contains a square term, then in accordance with the case (1) in the method of calculation; If ϕ does not contain a square term, then continue to follow the above steps, until the quadratic form is converted into the standard form..

(3) If $a_{11}=a_{12}=\Lambda=a_{1n}=0$, then we know $a_{21}=a_{21}=\Lambda=a_{n1}=0$ by symmetry, then the given quadratic form has $n-1$ variables, which can be operated by (1) or (2).

Example 3. Transforms quadratic form

$$f(x_1,x_2,\Lambda,x_n)=-4x_1x_2+2x_1x_2+2x_2x_3$$

into standard form.

Solution. Since $a_{11}=0\,(i=1,2,3)$, by calculation we get

$$f_1=\frac{1}{2}\frac{\partial f}{\partial x_1}=\frac{1}{2}(-4x_2+2x_3)=-2x_2+x_3,$$

$$f_2=\frac{1}{2}\frac{\partial f}{\partial x_2}=\frac{1}{2}(-4x_1+2x_3)=-2x_1+x_3.$$

So

$$f(x_1,x_2,\Lambda,x_n)=\frac{1}{a_{12}}\left[(f_1+f_2)^2-(f_1-f_2)^2\right]+\phi$$

$$-\frac{1}{4}\left[(2x_1+2x_2-2x_3)^2-(2x_1-2x_2)^2\right]+x_1^2$$

$$=-\left[(x_1+x_2-x_3)^2-(x_1-x_2)^2\right]+x_1^2.$$

Let

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$$\begin{cases} y_1 = x_1, \\ y_2 = x_1 + x_2 - x_3, \\ y_3 = x_1 - x_2, \end{cases}$$

that is,

$$\begin{cases} x_1 = y_1, \\ x_2 = -y_1 + y_3, \\ x_3 = 2y_1 - y_2 - y_3, \end{cases}$$

or $X = CY$, where

$$C = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix},$$

then $f(x_1, x_2, \Lambda, x_n) = y_1^2 - y_2^2 + y_3^2$.

IV. CONCLUSION

Undergraduates majoring in mathematics need to master and flexibly apply the above-mentioned methods, especially the basic methods and the orthogonal transformation method, and be able to synthesize linear Algebra and analytic geometry to solve mathematical problems flexibly.

In the process of transforming real quadratic form into standard form by orthogonal transformation method, how to obtain orthogonal matrix is a difficult point and a comprehensive method, which can help us to use the knowledge of Matrix and linear transformation synthetically, is Conducive to the knowledge of the mastery. Using partial derivative method to change quadratic form into standard form does not need to observe quadratic form to formulate, but is a standardized method. In addition, the process of solution is simpler and easier to understand than the methods of collocation, orthogonal transformation and elementary transformation (contract transformation) .

To understand and deeply understand the essence and skills of the methods of changing quadratic form into standard form, we can solve the problems of simplifying quadratic form quickly and correctly, and can be used to solve some practical problems.

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REFERENCES

[1] Former Algebra Group, department of Geometry and Algebra, department of Mathematics, Peking University, third edition. BEIJING: Higher Education Press, 2003(3) : 201-220.
[2] X.Y. Ma, and L. Fang, Selected lectures on higher algebra, Tianjin Education Press, 2010.
[3] J.L. Qian, D.Y Liu, The essence of solving advanced algebra problems. BEIJING: University Press, 2005.234-258.
[4] W.H. Chen, A special method of transforming quadratic form into standard form. Journal of Anhui Radio and Television University, 2001(3) : 13-15.
[5] J. Bai, A new method of transforming quadratic form into standard form. Department of Education, Taiyuan University. 26(2) : 111-112,2008.
[6] J. Liu, An elementary transformation method of transforming quadratic form into standard form by orthogonal transformation. Journal of Liaoning Institute of Technology. 20(3) : 64-65,2000.
[7] S.Z. Li, Methods and techniques for advanced Algebra. 2004.

Author's biography with Photo

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