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Application examples of Vandermonde determinant in linear space and linear transformation

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***Abstract*—Vandermonde determinant is a special determinant, which is widely used, not only in the calculation of determinant, but also in linear space, linear transformation and so on. Vandermonde determinant is a very famous determinant because of its special form, clear calculation results and easy identification. How to transform determinant into Vandermonde determinant is a difficult and important problem. Based on the brief introduction of Vandermonde determinant, this paper illustrates its application in linear space and linear transformation.**

***Index Terms*—Vandermonde determinant, linear space, linear transformation, determinant, matrix.**

I. Introduction

Vandermonde determinant was discovered by the French mathematician Vandermonde in 1772. It is the following determinant of order n

$$d=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & a_3^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}. \qquad (1)$$

For any n, the value of Vandermonde determinant of order n is equal to $\prod\limits_{1\le j<i\le n}(a_i-a_j)$. This result can be proved by mathematical induction or recurrence method, and will not be repeated here.

It is easy to know from the value of Vandermonde determinant that Vandermonde determinant is equal to zero if and only if at least two of them are equal.

Vandermonde determinant is a very famous determinant because of its special form, clear calculation results and easy identification. Vandermonde determinant is widely used, not only in the calculation of determinant, but also in linear space and linear transformation [1-6]. This paper mainly illustrates its application in linear space and linear transformation by concrete examples.

II. Application of Vandermonde Determinant in Linear Space

In linear space, we often encounter the problem of

transformation using Vandermonde determinant. Through transformation, we can easily get the conclusion we need.

Example 1. Let V be a n dimensional linear space over the number field P. it is proved that for any positive integer m $(m\ge n)$, there are m vectors in P, and any n vectors are linearly independent.

Proof. Since V and P^n are isomorphic, we only need to consider them in P^n.

Let

$$a_1=(1,2,2^2,\cdots,2^{n-1}),$$
$$a_2=(1,2^2,(2^2)^2,\cdots(2^2)^{n-1}),$$
$$\cdots\cdots$$
$$a_m=(1,2^m,(2^m)^2,\cdots(2^m)^{n-1}),$$

and

$$D_n=\begin{vmatrix} 1 & 2^{k_1} & (2^{k_1})^2 & \cdots & (2^{k_1})^n \\ 1 & 2^{k_2} & (2^{k_2})^2 & \cdots & (2^{k_2})^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 2^{k_n} & (2^{k_n})^2 & \cdots & (2^{k_n})^n \end{vmatrix},$$

where $1\le k_1<k_2<\cdots<k_n\le m$, then D_n is a Vandermonde determinant and $D_n\ne 0$, so $a_{k_1},a_{k_2},\cdots,a_{k_n}$ are linearly independent.

Example 2. $P^{2\times2}$ is used to represent the set of all 2 order matrices over P. Let a_1,a_2,a_3,a_4 be four number that differs from each other and $a_1+a_2+a_3+a_4\ne0$. Proved that

$$A_1=\begin{pmatrix} 1 & a_1 \\ a_1^2 & a_1^4 \end{pmatrix}, A_2=\begin{pmatrix} 1 & a_2 \\ a_2^2 & a_2^4 \end{pmatrix},$$

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$$A_3=\begin{pmatrix}1 & a_3\\ a_3^2 & a_3^4\end{pmatrix}, A_4=\begin{pmatrix}1 & a_4\\ a_4^2 & a_4^4\end{pmatrix}$$

are a set of bases of linear spaces $P^{2\times 2}$ on P.

Proof. Let $x_1,x_2,x_3,x_4\in P$ and

$$x_1A_1+x_2A_2+x_3A_3+x_4A_4=0,$$

then

$$\begin{pmatrix}x_1 & a_1x_1\\ a_1^2x_1 & a_1^4x_1\end{pmatrix}+\begin{pmatrix}x_2 & a_2x_2\\ a_2^2x_2 & a_2^4x_2\end{pmatrix}+\begin{pmatrix}x_3 & a_3x_3\\ a_3^2x_3 & a_3^4x_3\end{pmatrix}$$
$$+\begin{pmatrix}x_4 & a_4x_4\\ a_4^2x_4 & a_4^4x_4\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

So

$$\begin{cases}x_1+x_2+x_3+x_4=0\\ a_1x_1+a_2x_2+a_3x_3+a_4x_4=0\\ a_1^2x_1+a_2^2x_2+a_3^2x_3+a_4^2x_4=0\\ a_1^4x_1+a_2^4x_2+a_3^4x_3+a_4^4x_4=0\end{cases}$$

i.e.,

$$\begin{pmatrix}1 & 1 & 1 & 1\\ a_1 & a_2 & a_3 & a_4\\ a_1^2 & a_2^2 & a_3^2 & a_4^2\\ a_1^4 & a_2^4 & a_3^4 & a_4^4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\\ 0\end{pmatrix}.$$

Let the coefficient determinant of the linear homogeneous equations be

$$D_4=\begin{vmatrix}1 & 1 & 1 & 1\\ a_1 & a_2 & a_3 & a_4\\ a_1^2 & a_2^2 & a_3^2 & a_4^2\\ a_1^4 & a_2^4 & a_3^4 & a_4^4\end{vmatrix},$$

and add one row and one column to D_4 to become the Vandermonde determinant

$$D_5(y)=\begin{vmatrix}1 & 1 & 1 & 1 & 1\\ a_1 & a_2 & a_3 & a_4 & y\\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & y^2\\ a_1^3 & a_2^3 & a_3^3 & a_4^3 & y^3\\ a_1^4 & a_2^4 & a_3^4 & a_4^4 & y^4\end{vmatrix}$$

$$=(y-a_1)(y-a_2)(y-a_3)(y-a_4)\prod_{1\le j<i\le 4}(a_i-a_j).$$

Since D_4 is the opposite number of the coefficient of y^3 in $D_5(y)$, and from the right side of the above formula, we know that the coefficient of y^3 is

$$(-\sum_{i=1}^{4}a_i)\prod_{1\le j<i\le 4}(a_i-a_j),$$

so

$$D_4=(\sum_{i=1}^{4}a_i)\prod_{1\le j<i\le 4}(a_i-a_j).$$

According to $\sum_{i=1}^{4}a_1\neq 0, a_i\neq a_j(i\neq j)$, we have $D_4\neq 0$, so the above linear equations have only zero solutions. Therefore, A_1,A_2,A_3,A_4 are linearly independent and form a set of bases of $P^{2\times 2}$.

III. Application of Vandermonde Determinant in Linear Transformation

Example 3. Let the linear transformation σ of n dimensional linear space V over the number field P have n distinct eigenvalues $\lambda_1,\lambda_2,\Lambda,\lambda_n$, then

(1) The linear transformation of V exchangeable with σ is the linear combination of $\varepsilon,\sigma,\sigma^2,\Lambda,\sigma^{n-1}$, where ε is the identity transformation.

(2) $\forall\alpha\in V,\alpha,\sigma\alpha,\sigma^2\alpha,\Lambda,\sigma^{n-1}\alpha$ are linearly independent if and only if $\alpha=\sum_{i=1}^{n}\alpha_i$, where $\sigma(\alpha_i)=\lambda_i\alpha_i, i=1,2,\Lambda,n$.

Proof. Let δ be a linear transformation commutative with σ, and

$$\sigma(\alpha_i)=\lambda_i\alpha_i, i=1,2,\Lambda,n,$$

then $V_{\lambda_1}=\{k\alpha_1|k\in P\}$ is the invariant subspace of δ. Let

$$\delta=x\varepsilon+x_1\sigma+x_2\sigma^2+\Lambda+x_{n-1}\sigma^{n-1}$$

and $\delta(\alpha_i)=k_i\alpha_i, i=1,2,\Lambda,n$, then we have

$$\begin{cases}k_1=x+x_1\lambda_1+x_2\lambda_1^2+\Lambda+x_{n-1}\lambda_1^{n-1}\\ k_2=x+x_1\lambda_2+x_2\lambda_2^2+\Lambda+x_{n-1}\lambda_2^{n-1}\\ \Lambda\ \Lambda\ \Lambda\ \Lambda\ \Lambda\ \Lambda\ \Lambda\ \Lambda\\ k_n=x+x_1\lambda_n+x_2\lambda_n^2+\Lambda+x_{n-1}\lambda_n^{n-1}.\end{cases}\qquad(2)$$

Since the coefficient determinant of the system (2) is Vandermonde determinant and

$$D=\prod_{1\le j<i\le n}(\lambda_i-\lambda_j),$$

the system (2) has a unique solution, so δ is the linear combination of $\varepsilon,\sigma,\sigma^2,\Lambda,\sigma^{n-1}$.

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(2) Sufficiency. Since $\alpha=\sum_{i=1}^{n}\alpha_i$, so we have

$$(\alpha,\sigma(\alpha),\cdots,\sigma^{n-1}(\alpha))$$
$$=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{vmatrix}1 & \lambda_1 & \cdots & \lambda_1^{n-1}\\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_n & \cdots & \lambda_n^{n-1}\end{vmatrix},$$

and

$$\begin{vmatrix}1 & \lambda_1 & \cdots & \lambda_1^{n-1}\\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_n & \cdots & \lambda_n^{n-1}\end{vmatrix}=\prod_{1\le j<i\le n}(\lambda_i-\lambda_j)\neq 0.$$

Therefore,

$$\begin{pmatrix}1 & \lambda_1 & \cdots & \lambda_1^{n-1}\\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1}\\ \vdots & \vdots & & \vdots\\ 1 & \lambda_n & \cdots & \lambda_n^{n-1}\end{pmatrix}$$

is an invertible matrix, and since $\alpha_1,\alpha_2,\cdots,\alpha_n$ is a set of bases of V, $\alpha,\sigma(\alpha),\cdots,\sigma^{n-1}(\alpha)$ is also a set of bases of V, they are linearly independent.

Necessity. Let $e_1,e_2,\cdots,e_n$ be the eigenvectors of σ corresponding to eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ respectively, then $e_1,e_2,\cdots,e_n$ form a base of V, so $\alpha=k_1e_1+k_2e_2+\cdots+k_ne_n$.

If $k_i\neq 0, i=1,2,\cdots,n$, then k_ie_i is also the eigenvector of σ corresponding to eigenvalue λ_i, so the conclusion is true.

If there exists $j\in\{1,2,\cdots,n\}$ and $k_j\neq 0$, without losing generality, we assume $k_1,k_2,\cdots,k_r$ are not be zero, and $k_{r+1}=\cdots=k_n=0$, then

$$\alpha=k_1e_1+k_2e_2+\cdots+k_re_r.$$

Therefore, we have

$$(\alpha,\sigma(\alpha),\cdots,\sigma^{n-1}(\alpha))$$
$$=(e_1,e_2,\cdots,e_n)\begin{vmatrix}k_1 & k_1\lambda_1 & \cdots & k_1\lambda_1^{n-1}\\ k_2 & k_2\lambda_2 & \cdots & k_2\lambda_2^{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ k_r & k_r\lambda_r & \cdots & k_r\lambda_r^{n-1}\end{vmatrix}$$
$$=(e_1,e_2,\cdots,e_r)A.$$

Using the Vandermonde determinant, we can know that A has a r order subdeterminant that is not zero, so $\operatorname{rank}(A)=r$, thus

$$\operatorname{rank}(\alpha,\sigma(\alpha),\cdots,\sigma^{n-1}(\alpha))=r,$$

and since $r<n$, so $\alpha,\sigma(\alpha),\cdots,\sigma^{n-1}(\alpha)$ are linearly related, which is contradictory to the condition of the problem. So $\alpha=\sum_{i=1}^{n}\alpha_i$, where

$$\sigma(\alpha_i)=\lambda_i\alpha_i, i=1,2,\cdots,n.$$

Example 4. Let V be a n dimensional linear space over the number field P, σ,τ be two linear transformations of V, and σ has n distinct eigenvalues on P, then

(1) The eigenvectors of σ are eigenvectors of τ if and only if $\sigma\tau=\tau\sigma$;

(2) If $\sigma\tau=\tau\sigma$, then τ is a linear combination of $\varepsilon,\sigma,\sigma^2,\cdots,\sigma^{n-1}$, where ε is the identity transformation on V.

Proof. Let $\lambda_1,\lambda_2,\cdots,\lambda_n$ be the different eigenvalues of σ, and $\alpha_1,\alpha_2,\cdots,\alpha_n$ be the eigenvectors of σ corresponding to the eigenvalues $\lambda_1,\lambda_2,\cdots,\lambda_n$ respectively. Since $\lambda_1,\lambda_2,\cdots,\lambda_n$ are different from each other, so $\alpha_1,\alpha_2,\cdots,\alpha_n$ are a set of bases of linear space V.

(1) Necessity. If each α_i is an eigenvector of τ, then there exists $\mu_i\in P$ such that $\tau(\alpha_i)=\mu_i\alpha_i, i=1,\cdots,n$. So

$$\sigma\tau(\alpha_i)=\sigma(\mu_i\alpha_i)=\mu_i\sigma(\alpha_i)=\mu_i\lambda_i\alpha_i$$
$$=\lambda_i(\mu_i\alpha_i)=\lambda_i(\tau(\alpha_i))=\tau(\lambda_i\alpha_i)$$
$$=\tau(\sigma(\alpha_i))=\tau\sigma(\alpha_i), i=1,2,\cdots,n.$$

Since $\alpha_1,\alpha_2,\cdots,\alpha_n$ are a group of bases of V, so $\sigma\tau=\tau\sigma$.

Sufficiency. Let $\sigma\tau=\tau\sigma$. For every α_i $(i=1,2,\cdots,n)$, we have

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$$\sigma(\tau(\alpha_i))=(\sigma\tau)\alpha_i=(\tau\sigma)\alpha_i$$
$$=\tau(\sigma(\alpha_i))=\tau(\lambda_i\alpha_i)=\lambda_i(\tau(\alpha_i)).$$

So $\tau(\alpha_i)\in V_{\lambda_i}$. Since $\dim(V_{\lambda_i})=1$, there is $\mu_i\in P$ so that $\tau(\alpha_i)=\mu_i\alpha_i, i=1,2,\cdots,n$. So α_i is also the eigenvector of τ, $i=1,2,\cdots,n$. So the eigenvector of σ is also the eigenvector of τ.

(2) Let $\sigma\tau=\tau\sigma$, from (1) we know that α_i is also the eigenvector of τ, $i=1,2,\cdots,n$, so there exists $\mu_i\in P$ such that $\tau(\alpha_i)=\mu_i\alpha_i, i=1,2,\cdots,n$, then we get

$$\sigma(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix},$$

$$\tau(\alpha_1,\alpha_2,\cdots,\alpha_n)=(\alpha_1,\alpha_2,\cdots,\alpha_n)\begin{bmatrix}\mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n\end{bmatrix}.$$

Consider the linear equations

$$\begin{cases} x_1+\lambda_1x_2+\cdots+\lambda_1^{n-1}x_n=\mu_1 \\ x_1+\lambda_2x_2+\cdots+\lambda_2^{n-1}x_n=\mu_2 \\ \qquad\cdots\cdots \\ x_1+\lambda_nx_2+\cdots\ \lambda_n^{n-1}x_n=\mu_n. \end{cases} \quad (3)$$

Since the coefficient determinant

$$\begin{vmatrix}1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-1}\end{vmatrix}=\prod_{1\le j<i\le n}(\lambda_i-\lambda_j)\neq 0.$$

Then the system of equations (3) has a unique solution $(a_0,a_1,\cdots,a_{n-1})'$, thus

$$a_0+a_1\lambda_i+\cdots+a_{n-1}\lambda_i^{n-1}=\mu_i, i=1,2,\cdots,n.$$

So

$$(a_0+a_1\lambda_i+\cdots+a_{n-1}\lambda_i^{n-1})\alpha_i=\mu_i\alpha_i, i=1,2,\cdots,n.$$

Since $\sigma(\alpha_i)=\lambda_i\alpha_i, \tau(\alpha_i)=\mu_i\alpha_i$, so

$$(a_0\varepsilon+a_1\sigma+a_2\sigma^2+\cdots+a_{n-1}\sigma^{n-1})\alpha_i=\tau(\alpha_i),$$
$$i=1,2,\cdots,n.$$

Taking into accout $\alpha_1,\alpha_2,\cdots,\alpha_n$ is a group of bases of V, we obtain $\tau=a_0\varepsilon+a_1\sigma+a_2\sigma^2+\cdots+a_{n-1}\sigma^{n-1}$.

IV. CONCLUSION

The calculation result of Vandermonde determinant is very concise. The difficulty is how to connect the given determinant with Vandermonde determinant and turn it into the form of Vandermonde determinant. This requires a relatively high observation and calculation skills, need to pay attention to observation in the process of learning, constantly summarize the rules and methods, improve problem-solving skills. The application of Vandermonde determinant can be accomplished by mastering the applicable forms and skills of Vandermonde determinant. The application examples of Vandermonde determinant in linear space and linear transformation in this paper reflect the integration and penetration of Vandermonde determinant and other mathematical knowledge, which contains high problem-solving skills. How to skillfully and flexibly use Vandermonde determinant to solve practical problems and apply it is worth further exploring.

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Author's biography with Photo

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