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A New Family of Newton-type Methods with Cubic Convergence

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***Abstract*—In this paper, we present a new class of third-order iterative methods, which are variants of the Newton's method. In contrast to the Newton's method, the presented methods are more effective and numerical examples are given to show the efficiency.**

***Key words*—Newton's method, third-order convergence, nonlinear equations.**

I. INTRODUCTION

We consider the iterative methods for finding a simple root α of a nonlinear equation $f(x)=0$, where $f: I \subseteq R \to R$ for an open interval I is a scalar function. The well-known and widely used method is the classical Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which converges quadratically in some neighborhood of α.

In recent years, many modifications of Newton's method with at least cubic convergence have been proposed, see[1-14]and references therein. Many numerical applications use high precision in their computation, so higher-order numerical methods are important[15]. In this paper, we first present a new class of iterative methods with third-order convergence. Then, numerical examples are given to show the performance.

II. CONVERGENCE ANALYSIS

Now, we consider the iteration scheme

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \dfrac{f(x_n)+(C+1)f(y_n)}{f(x_n)+Cf(y_n)} \dfrac{f(x_n)}{f'(x_n)} \end{cases} \tag{2}$$

which is a variant of the Newton's method and where C is an arbitrary constant.

Theorem Let α be a simple zero of sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the method defined by (2) is of third-order and satisfies the error equation

$$e_{n+1} = -(C+2)e_n^3 + O(e_n^4),$$

where $e_n = x_n - \alpha$ and $c_k = f^{(k)}(\alpha)/k!f'(\alpha)$.

Proof Using Taylor expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)], \tag{3}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)]. \tag{4}$$

(4)

Furthermore, we can get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3)e_n^3 + O(e_n^4), \tag{5}$$

$$f(y_n) = f'(\alpha)[c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4)], \tag{6}$$

and

$$\begin{aligned} f(x_n) + Cf(y_n) = f'(\alpha)[e_n + (C+1)c_2 e_n^2 \\ + \left(c_3 - 2C(c_2^2 - c_3)\right)e_n^3 + O(e_n^4)], \end{aligned} \tag{7}$$

$$\begin{aligned} f(x_n) + (C+1)f(y_n) = f'(\alpha)[e_n + (C+2)c_2 e_n^2 + \\ \left(c_3 - 2(C+1)(c_2^2 - c_3)\right)e_n^3 + O(e_n^4)] \end{aligned}. \tag{8}$$

From (5,7,8), we obtain

$$e_{n+1} = -(C+2)e_n^3 + O(e_n^4).$$

This means the method defined by (2) is of third-order. That completes the proof.

Especially, if $C=-2$, we obtain a fourth-order method

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = x_n - \dfrac{f(x_n)-f(y_n)}{f(x_n)-2f(y_n)} \dfrac{f(x_n)}{f'(x_n)} \end{cases},$$

which is the Ostrowski's method.

III. NUMERICAL EXAMPLES

In this section, we employ the new methods defined by (2) to solve some nonlinear equations and compare them with Newton's method (NM). C is an arbitrary constant and here wo let $C=-3/2$. Displayed in Table I are the number of iterations (IT) and the number of function evaluations (NFE) required such that $|f(x_n)| < 10^{-15}$.

We use the following functions:

$$f_1(x) = x^3 + 4x^2 - 10,$$
$$\alpha = 1.3652300134140968879137 3,$$
$$f_2(x) = x^5 + x^4 + 4x^2 - 20,$$
$$\alpha = 1.4662790738647226707058 7,$$
$$f_3(x) = x^3 - e^{-x},$$
$$\alpha = 0.7728829591492101247496 29,$$
$$f_4(x) = e^x \sin x + \ln(x^2+1),$$
$$\alpha = 0.$$

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The computational results presented in Table 1 show that, the presented methods converge more rapidly than Newton's method and require the less NFE. Therefore, the new methods (2) have better convergence efficiency.

Table I: Comparison of various iterative methods

	x_0	IT(NM)	NFE(NM)	IT(Eq.(2))	NFE(Eq.(2))
	-1	24	48	19	57
$f_1(x)$	1	5	10	3	9
	2	5	10	3	9
	0.5	10	20	4	12
$f_2(x)$	1.2	5	10	3	9
	2	6	12	4	12
	0	6	12	4	12
$f_3(x)$	0.5	5	10	3	9
	-0.5	7	14	4	12
	-1	8	16	3	9
$f_4(x)$	0.5	7	14	4	12
	2	7	14	4	12

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