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Application of Vector in Geometric Proof

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***Abstract*—Vector is an essential tool to integrate intuitive graphics and algebra and connect geometry and algebra. It is the unity of form and number. Sometimes it may be very complicated to solve some geometric problems with conventional geometric proof methods, but using vectors to transform geometric problems into algebraic operations will often simplify the problem-solving process and make it easier to understand, which undoubtedly reflects the idea of the combination of numbers and shapes in mathematics. Through some typical examples, this paper discusses how to solve the problems encountered in geometric proof with vector method, reflects the importance of these methods in proving geometric problems, then summarizes some skills in solving problems, understands the thinking mode of vector problem-solving, and provides broader ideas for future research and learning.**

***Index Terms*—Vector, geometric proof, algebraic operation, combination of number and shape, mode of thinking.**

I. Introduction

Vector is a concept widely used in the fields of mathematics and physics. The quantities with both size and direction such as displacement, velocity and force that can be seen everywhere in real life are its physical background, and the directed line segment is its geometric background [1-7]. Vector is a mathematical concept abstracted from these practical objects. After studying and establishing a complete knowledge system, it has been widely used to solve problems in mathematical geometry, physics and real life. Therefore, its position in mathematics is self-evident. In addition, using various properties and algorithms of vectors to solve elementary mathematical problems, especially geometric problems, can save many complex steps, and the process is simple, standardized and clear. The conclusion obtained by the same method can not only solve the problems related to graphics in two-dimensional linear space, but also solve the corresponding problems in three-dimensional linear space, which is another important feature of applying vector to solve problems. Vector is not only an important tool to algebraic geometric structure, but also an effective method of geometric proof. Therefore, the correct application of vector algebra knowledge combined with certain mathematical ideas and methods can greatly reduce the difficulty of solving problems, which is a very important application skill of vector in geometric proof. This paper will discuss and summarize the application skills of vector in geometric proof by analyzing and exploring specific examples and drawing lessons from relevant literature.

II. Using vector to solve the problem of line coplanarity

A. *Using the vertical property of vector and vector decomposition, it is proved that lines are common*

Definition 1 The coplanar problem of lines means that two or more nonparallel lines intersect at a point, then these lines are called coplanar.

By using the vertical property of vector and vector decomposition, geometric graphics can be transformed into algebraic operations, which can simplify geometric problems.

Example 1. In ΔABC , the heights on the side of AB, AC, BC are CD, BF, AE respectively. Verification: CD, BF, AE intersect at a point G (as shown in Figure 1).

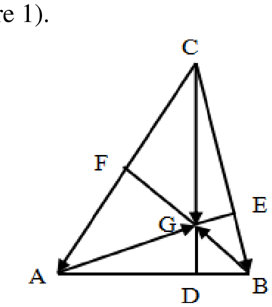

Figure 1

Proof. Let the high BF, AE on the edge of AC and BC intersect at a point G and connect CG .

Since $AC \perp BF$, then $\vec{AC}\cdot\vec{BF}=0$, i.e.,, $\vec{AC}\cdot\vec{BG}=0$. The vector $\vec{BG}$ is decomposed into $\vec{BG}=\vec{CG}-\vec{CB}$, then

$$\vec{AC}\cdot\vec{BG}=\vec{AC}\cdot(\vec{CG}-\vec{CB})=0,$$

i.e.,

$$\vec{AC}\cdot\vec{CG}=\vec{AC}\cdot\vec{CB}.$$

Similarly , since $BC \perp AE$, so $\vec{BC}\cdot\vec{AE}=0$,i.e.,

$\vec{BC}\cdot\vec{AG}=0$. The vector $\vec{AG}$ is decomposed into $\vec{AG}=\vec{CG}-\vec{CA}$, then

$$\vec{BC}\cdot\vec{AG}=\vec{BC}\cdot(\vec{CG}-\vec{CA})=0.$$

That is $\vec{BC}\cdot\vec{CG}=\vec{BC}\cdot\vec{CA}$.

Since $\vec{AC}\cdot\vec{CB}=\vec{BC}\cdot\vec{CA}$, so $\vec{AC}\cdot\vec{CG}=\vec{BC}\cdot\vec{CG}$, i.e.,

$$\vec{BC}\cdot\vec{CG}-\vec{AC}\cdot\vec{CG}=(\vec{BC}-\vec{AC})\cdot\vec{CG}$$
$$=\vec{BA}\cdot\vec{CG}=0.$$

So, $BA\perp CG$. Thus point G is on the height CD of the side AB . Therefore, CD,BF,AE intersect at the point G . The proof is complicted.

The key to prove this type of problem is to find the appropriate vector decomposition form and prove it by using the vertical property of vector. The appropriate use of vector knowledge is the key to prove geometry problems.

B. Using the combination of vector theory and undetermined coefficient method to prove that the lines are common

Example 2. As shown in Figure 2, in the ΔABC , D,E,F are the midpoint of BC,AC,AB . Prove that three midlines AD,BE,CF intersect at the point H .

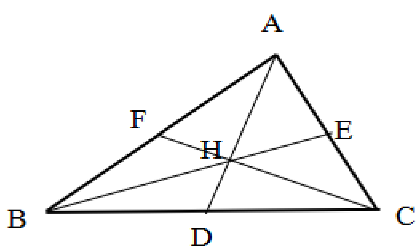

Figure 2

The purpose of this problem is to prove that the three midlines of any triangle are in common, and the intersection is called the center of gravity. The combination of vector algebra theory and undetermined coefficient method is used to prove that this problem can reduce the difficulty of solving the problem.

Proof. As shown in Figure 3, we select two line segments as a group of bases $\vec{BA},\vec{BC}$. Let H_1 be a point on the the center line BE, and $\vec{BH_1}=\lambda_1\vec{H_1E}$, H_2 be a point on the the center line CF, and

$$\vec{CH_2}=\lambda_2\vec{H_2F},(\lambda_1,\lambda_2\neq 0).$$

Since

$$\vec{BE}=\vec{BH_1}+\vec{H_1E}$$
$$=\vec{BH_1}+\frac{1}{\lambda_1}\vec{BH_1}=\frac{\lambda_1+1}{\lambda_1}\vec{BH_1},$$
$$\vec{BC}=\vec{BH_2}-\vec{CH_2}=\vec{BH_2}-\lambda_2\vec{H_2F}$$
$$=\vec{BH_2}-\lambda_2(\vec{BF}-\vec{BH_2})=(\lambda_2+1)\vec{BH_2}-\lambda_2\vec{BF}.$$

So we have

$$\vec{BH_1}=\frac{\lambda_1}{\lambda_1+1}\vec{BE}=\frac{\lambda_1}{2(\lambda_1+1)}(\vec{BC}+\vec{BA})$$
$$=\frac{\lambda_1}{2(\lambda_1+1)}\vec{BC}+\frac{\lambda_1}{2(\lambda_1+1)}\vec{BA},$$
$$\vec{BH_2}=\frac{\vec{BC}+\lambda_2\vec{BF}}{\lambda_2+1}$$
$$=\frac{2}{2(\lambda_2+1)}\vec{BC}+\frac{\lambda_2}{2(\lambda_2+1)}\vec{BA}.$$

Let $\vec{BH_1}=\vec{BH_2}$, then $\lambda_1=\lambda_2=2$. Therefore, the center lines BE and CF intersect at point H .

Similarly, it can be proved that the midline BD also passes through the H point, so the original proposition can be proved.

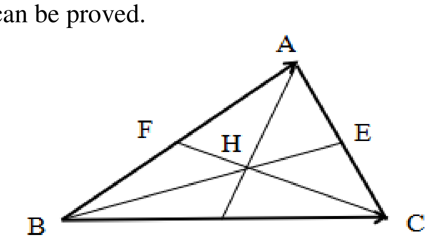

Figure 3

To solve this type of problem, we need to properly introduce undetermined parameters according to the problem setting, and then use the vector algebra theory and problem setting conditions to further calculate the parameters, so that the problem can be proved and the problem can be solved.

III. Application of Vector Product and Quantity Product of Vector in Geometric Proof

A. Using the vector product of vectors to prove that the surface is vertical

Definition 2. The vector product (also known as the outer product) of two vectors is a vector, recorded as $\vec{a}\times\vec{b}$, and its module is $\left|\vec{a}\times\vec{b}\right|=\left|\vec{a}\right|\left|\vec{b}\right|\sin\angle(a,b)$.

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Theorem 1 It is known that the normal vectors of plane α and plane β are $\vec{n}=(x_1,y_1,z_1)$ and $\vec{m}=(x_2,y_2,z_2)$ respectively. If $\vec{n}\perp\vec{m}$, that is, $\vec{n}\cdot\vec{m}=0$, then plane α is perpendicular to plane β, i.e., $\alpha\perp\beta$.

Establishing an appropriate space rectangular coordinate system, writing out the coordinates of each point and expressing each vector, which can be proved by using the vector product of vector and the judgment theorem of surface perpendicularity, which can greatly reduce the difficulty of solving the problem.

Example 3. In cube $ABCD-EFGH$, O and P are the midpoint of BF, DC respectively. Prove that the p;lanes ADO and EHP are perpendicular, i.e., $ADO\perp EHP$.

Proof. As shown in Figure 4, take D as the coordinate origin, and the straight lines where DA, DC, DH are located are axes x, y and z respectively. If the edge length of the cube is 1, we can easily get

$$D(0,0,0), A(1,0,0), O(1,1,\frac{1}{2}), P(0,\frac{1}{2},0),$$

$$E(1,0,1), H(0,0,1), B(1,1,0), F(1,1,1).$$

So,

$$\vec{DA}=(1,0,0),\quad \vec{HE}=(1,0,0),$$

$$\vec{HP}=(0,\frac{1}{2},-1),\quad \vec{DO}=(1,1,\frac{1}{2}).$$

Let the normal vectors of plane ADO and plane EHP be $\vec{n},\vec{m}$ respectively, then

$$\vec{n}=\vec{DA}\times\vec{DO}=\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 1 & 1 & \frac{1}{2} \end{vmatrix}$$

$$=-\frac{1}{2}\vec{j}+\vec{k}=(0,-\frac{1}{2},1),$$

$$\vec{m}=\vec{HP}\times\vec{HE}=\begin{vmatrix} i & j & k \\ 0 & \frac{1}{2} & -1 \\ 1 & 0 & 0 \end{vmatrix}$$

$$=\vec{j}+\frac{1}{2}\vec{k}=(0,1,\frac{1}{2}).$$

Since $\vec{n}\cdot\vec{m}=0$, so $\vec{n}\perp\vec{m}$, i.e., $ADO\perp EHP$.

The normal vector coordinates are obtained by using the vector product operation of two vectors, and then the normal vector vertical is obtained by using the quantity product operation of two vectors. Finally, it is proved by using the judgment theorem. The key to solving this problem is to master the operation law of vector product and use it correctly.

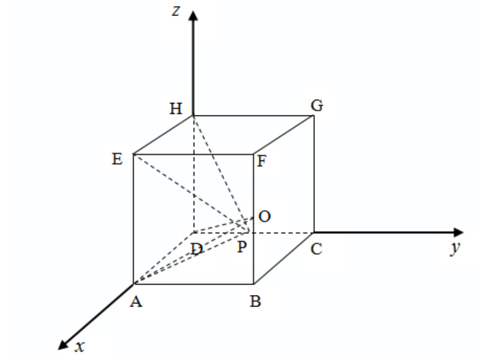

Figure 4

B. Proving the formula of trigonometric function by using the quantity product of vector

Definition 3. The quantitative product of two vectors $\vec{a},\vec{b}$ refers to the product of the modulus of two vectors and the cosine of their included angle, which is recorded as $\vec{a}\cdot\vec{b}$, that is,

$$\vec{a}\cdot\vec{b}=|\vec{a}|\cdot|\vec{b}|\cos\angle(\vec{a},\vec{b}).$$

Establish a rectangular coordinate system, transform the problem into a vector problem, and prove it with the knowledge of the quantity product of the vector. The following is an example.

Example 4 Proves the cosine formula of two angular differences by using the quantity product of vectors,i.e.,

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta.$$

Proof. Establish a plane rectangular coordinate system, take the origin O as the center of the circle as the unit circle, and take two points A, B on the axis and half x axis respectively.

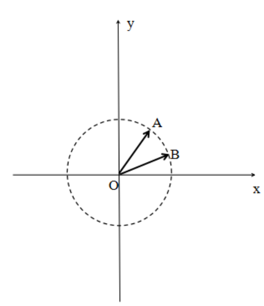

Figure 5

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Note that the included angle between $\vec{OA}$ and x axis positive half axis is α, and the included angle between $\vec{OB}$ and x axis positive half axis is β, and $\beta < \alpha$. Then

$$\vec{OA}=(\cos\alpha, \sin\alpha), \quad \vec{OB}=(\cos\beta, \sin\beta).$$

So we have $\vec{OA}\cdot\vec{OB} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

Using the quantity product formula of vector, we can get

$$\vec{OA}\cdot\vec{OB} = |\vec{OA}||\vec{OB}|\cos\angle(\vec{OA},\vec{OB})$$
$$= \cos\angle(\vec{OA},\vec{OB}) = \cos(\alpha-\beta).$$

Therefore we get

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

The proof is completed.

Using vector quantity product can not only prove space geometry problems, but also prove trigonometric function formula, convert trigonometric function into graphics, and then prove it by vector, which also reflects the mathematical idea of the combination of number and shape.

IV. SUMMARY OF THE SKILLS OF VECTOR IN GEOMETRIC PROOF

Through the analysis of the above different types of examples, it can be seen that using the relevant knowledge of vectors to prove geometric problems is usually relatively simple. It can save a lot of complicated steps, simplify the problem-solving process and make the idea clear. According to my own understanding of vector knowledge and my research on the above examples, I have summarized the following skills on the application of vectors in geometric proof:

(1) Be good at using rectangular coordinate system. Many solid geometry problems can be solved by establishing coordinate system and then using vector.

(2) Be good at proving geometric problems by using vector properties, vector decomposition and linear operation, and be familiar with vector properties and linear operation rules.

(3) Be good at using the quantity product and vector product of vector to solve problems, understand and flexibly use the formula of quantity product and vector product.

(4) In the process of doing geometric proof problems, we should cultivate the awareness of solving problems with vectors and the idea of combination and transformation of numbers and shapes.

(5) Using vector algebra theory combined with certain mathematical ideas, such as undetermined coefficient method, identity method, construction method, etc., is also a very important skill in geometric proof. In view of my limited ability, this paper only introduces an example of solving problems by using vector algebra theory and undetermined coefficient method.

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Authors' biography with Photo

Longfei Li was born in January 2000 in Tai'an, Shandong Province, China.. He is a 2018 undergraduate majoring in information and Computing Science (intelligent mobile development direction) in the College of Mathematics and Statistics of Taishan University. His research fields are mainly information and computer science and technology.

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