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Application of Orthogonal Matrix in Topology

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***Abstract*—In this paper, the definition of orthogonal matrix and some of its properties are studied. Combined with the research results of many excellent scholars at home and abroad, the application of orthogonal matrix in topology is further understood and discussed.**

***Keywords*—Orthogonal matrix; topology; standard orthogonal basis; transition matrix; topological group.**

I. Introduction

As an important concept in mathematics, matrix is the main research object of algebra, and also one of the useful tools to explore and apply mathematics. Orthogonal matrix is a very important and often used matrix in mathematics, which makes it have a very important position in matrix theory, and it can be widely used in other fields. In addition, there are some other properties of orthogonal matrix that do not have [1-6]. At present, there are many papers about orthogonal matrix, but most of them are about the properties of orthogonal matrix, but few of them are about the application of orthogonal matrix in various fields. This paper focuses on the common properties of orthogonal matrix and its application in topology.

II. Definition and Properties of Orthogonal Matrix

A. *Definition of orthogonal matrix*

In the course of advanced algebra, the definition of orthogonal matrix is involved in the discussion of the transformation formula from one set of standard orthogonal bases to another.

Let $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ and $\eta_1,\eta_2,\cdots,\eta_n$ be two groups of orthonormal bases of Euclidean space V, and the transition matrix between them is $A=(a_{ij})$, that is,

$$(\eta_1,\eta_2,\cdots,\eta_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Since $\eta_1,\eta_2,\cdots,\eta_n$ is a standard orthogonal basis, so we have

$$(\eta_i,\eta_j)=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases} \qquad (1)$$

Each column of matrix A is the coordinate of $\eta_1,\eta_2,\cdots,\eta_n$ under the standard orthogonal basis $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$. According to formula

$$(\alpha,\beta)=x_1y_1+x_2y_2+\cdots+x_ny_n=X'Y$$

and formula (1), it can be expressed as formula

$$a_{1i}a_{1j}+a_{2i}a_{2j}+\cdots+a_{ni}a_{nj}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases} \qquad (2)$$

which is equivalent to a matrix equation $A'A=E$, that is, $A^{-1}=A'$.

Definition 1. Let A be a real matrix of order n, if it satisfies $A'A=E$, then A is called an orthogonal matrix.

From the above discussion, we get that the transition matrix from one group of standard orthogonal bases to another group of standard orthogonal bases is an orthogonal matrix; on the contrary, if the first group of bases is a standard orthogonal basis and the transition matrix is an orthogonal matrix, then the second group of bases must also be a standard orthogonal basis.

B. *Properties of orthogonal matrix*

Property 1. The necessary and sufficient condition for a to be an orthogonal matrix is that all the column vectors (row vectors) of a are unit vectors, and the two are orthogonal.

Property 2. If A is an orthogonal matrix, then $|A|=\pm 1$.

Property 3. If A is an orthogonal matrix, then A^{-1} is also an orthogonal matrix.

Property 4. If A is an orthogonal matrix, then its adjoint matrix A^* is also an orthogonal matrix.

Proof. Since $A^*=|A|A^{-1}$, and $|A|=\pm 1$, so we have the following results.

(1) When $|A|=1$, we can get $A^*=A^{-1}$. From property 3, A^* is an orthogonal matrix.

(2) When $|A|=-1$, we can get

$$A^*=-A^{-1},(A^*)'A^*=(-A^{-1})'(-A^{-1})=(A^{-1})'A^{-1}=E.$$

Therefore, A^* is an orthogonal matrix.

Property 5. If A,B are both orthogonal matrices, then

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AB, $A'B, AB', A^{-1}B, AB^{-1}$ are also orthogonal matrices.

Property 6. If A is an orthogonal matrix of order n, then the modulus of the eigenvalue of A is 1.

Proof. Let x be a non-zero complex vector of n dimension, λ be a complex vector, and

$$Ax = \lambda x, \tag{3}$$

then

$$(\overline{Ax})' = (\overline{A}\overline{x})' = (\overline{x})'(\overline{A})' = (\overline{\lambda x})' = \overline{\lambda}\overline{x}'$$

is obtained by taking conjugate transposition at both ends of formula (3).

Since $A'A = E, \overline{x}'x > 0$, so

$$(\overline{x})'A'Ax = \overline{\lambda}\overline{x}'\lambda x, \overline{x}'x = \overline{\lambda}\lambda\overline{x}'x,,$$

that is $\overline{x}'x = |\lambda|^2 \overline{x}'x$. Therefore, we have $|\lambda|^2 = 1$. Thus, the modulus of the eigenvalue of A is 1.

III. THE APPLICATION OF ORTHOGONAL MATRIX IN TOPOLOGY

Because of some special properties of orthogonal matrix and its important position in matrix theory, orthogonal matrix has been widely used in many fields. In addition, we introduce the application of orthogonal matrix in topology.

The set of all n-order orthogonal matrices is written as $O_{(n)}$. If we look at algebra and topology, we can prove that $O_{(n)}$ constitutes a topological group, and further prove that $O_{(n)}$ is a compact Lie Group which is not connected.

Theorem 1. $O_{(n)}$ constitutes a topological group.

Before we prove that $O_{(n)}$ constitutes a topological group, let's introduce some related concepts.

Definition 2. Let X be a set and $\Im$ be a subset family of X. if $\Im$ satisfies the following conditions:

(1) $X, \phi \in \Im$;

(2) If $A, B \in \Im$, then $A \cap B \in \Im$;

(3) If $\Im_1 \subset \Im$, then $\cup_{A\in\Im_1} A \in \Im$,

then $\Im$ is a topology of X.

If $\Im$ is a topology of set X, then even pair $(X, \Im)$ is a topological space, or set X is a topological space relative to topology $\Im$.

Definition 3. If $\Im$ is a topological space, and gives its group structure, so that the following group multiplication and inversion operations:

multiplication $u: \Im \times \Im \to \Im$;

finding the inverse calculation $v: \Im \to \Im$,

are all continuous mappings, then $\Im$ is a topological group.

According to the above definition, we prove that the set $O_{(n)}$ of all n-order orthogonal matrices constitutes a topological group in the following three steps:

(1) $O_{(n)}$ constitutes a topological space;

(2) $O_{(n)}$ constitutes a group;

(3) $O_{(n)}$ constitutes a topological group.

Now prove theorem 1.

Proof. (1) Let R denote the set of all n-order matrices with real elements, and $A = (a_{ij})$ denote a representative element of R, then we can equate R with the n^2-dimensional Euclidean space G^{n^2}, that is to say, $A = (a_{ij})$ corresponds to the point $(a_{11}, a_{12}, \text{K}, a_{1n}, \text{K}, a_{n1}, \text{K}, a_{nn})$ of G^{n^2}. E is a subset family of point set G^{n^2}, so both G^{n^2} and ϕ belong to E. The union of any multiple sets in E belongs to E, and the intersection of finite sets in E also belongs to E. It can be verified that G^{n^2} constitutes a topological space, so R becomes a topological space. Because $O_{(n)}$ is an n-order orthogonal matrix of all real elements, so $O_{(n)}$ is a subset of R. Therefore, the topology of this subset can be induced by R, and $O_{(n)}$ forms a sub topological space of R.

(2) First of all, for any $A, B, C \in O_{(n)}$, since the multiplication of matrix satisfies the combination law, so $(AB)C \to A(BC)$.

Secondly, there exists $T_n \in O_{(n)}$, such that $T_n A = AT_n = A$.

Again, for any $A \in O_{(n)}$, there is $A^{-1} = A'$, which satisfies $A^{-1}A = A'A = AA^{-1} = AA' = E$. Therefore, the set $O_{(n)}$ made of orthogonal matrix can form a group for multiplication operation.

(3) For the topology of R in (1), define matrix multiplication

$$m: \quad R \times R \to R, \quad \forall\ A = (a_{ij}), B = (b_{ij}),$$

and the (i, j) elements of product $m(A, B)$ are $\sum_{k=1}^{n} a_{ik} b_{kj}$.

Now R is a topology with product space (n^2 factors). For any i, j satisfying $1 \le i, j \le n$, there is a projection map KKK, which maps the product

$$\Pi_{ij}\ m: \quad R \times R \to R \to E^1$$

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of A and B as its (i, j) element.

Since $\sum_{k=1}^{n} a_{ik} b_{kj}$ is a polynomial of the elements of A and B, so $\pi_{ij} m$ is continuous, and the projection mapping π_{ij} is continuous, and mapping m is continuous. Because $O_{(n)}$ is a subspace topology of R. According to the properties of orthogonal matrix and previous research, mapping $O_{(n)} \times O_{(n)} \to O_{(n)}$ is also continuous.

The matrix in $O_{(n)}$ is invertible, and $f : O_{(n)} \to O_{(n)}$, $\forall A \in O_{(n)}, f(A) = A^{-1}$ is defined. Since the composite mapping $\pi_{ij} f$: $O_{(n)} \to O_{(n)} \to E^1$ maps any $A \in O_{(n)}$ as the (i, j) element of A^{-1}, from the property $A' = \frac{A^*}{|A|}$ of orthogonal matrix, we can get $a_{ji} = \frac{A_{ji}}{|A|}$, that is, $\pi_{ij} f(A) = \frac{A_{ji}}{|A|}$. The determinant and the algebraic remainder of A are polynomials of the inner elements of A, and $|A| \neq 0$, so $\pi_{ij} f$ is continuous, so the projection f : $O_{(n)} \to O_{(n)}$ is continuous.

Therefore, $O_{(n)}$ is a topological space, and it forms a group. From the point of view of group multiplication and inversion, they are all continuous maps of topological space, so the set $O_{(n)}$ made by all n-order orthogonal matrices constitutes a topological group, which is called an orthogonal group.

Theorem 2. $O_{(n)}$ is a compact Lie group.

Before proving theorem 2, the definition and lemma related to it are introduced.

Definition 4. If G is a topological group, the topology of G is n-dimensional real or complex analytic manifold, and the mapping $(\sigma, \tau) \to \sigma\tau^{-1}, \forall \sigma, \tau \in G$ is the analytic mapping from the analytic manifold $G \times G$ to G, then G is a dimensional Lie group.

Lemma 1. Bounded closed sets in Euclidean spaces are compact subsets.

Now prove theorem 2.

Proof. For any $A \in R$ (set of all n-order matrices with real elements), A corresponds to the point $\alpha(a_{11}, a_{12}, \cdots, a_{1n}, \cdots, a_{n1}, \cdots, a_{nn})$ of n^2 dimension Euclidean space G^{n^2}, and R can be regarded as n^2 dimension Euclidean space. The determinant $\det A$ is the analytic function of element

$$a_{11}, a_{12}, \cdots, a_{1n}, \cdots, a_{n1}, \cdots, a_{nn},$$

and

$$R^* = \{A \in R \mid \det A = 0\}$$

is the open subset of R. According to the induced topology, R^* is an analytic manifold, and the multiplication and inversion operations of the matrix are all analytic, so R^* is an n^2 dimensional Lie group. $O_{(n)}$ is a closed subset of R^*, which is submanifold according to induced topology, and $O_{(n)}$ is lie group.

In order to prove the compactness of $O_{(n)}$, according to the theorem, if R is equal to G^{n^2}, $O_{(n)}$ is equal to the bounded closed set in G^{n^2}.

For any $A \in O_{(n)}$, because $A'A = E$, so $\sum_{j=1}^{n} a_{ij} b_{kj} = \delta_{ik} 1 \le i, k \le n$. For any i, k, define mapping

$$f_{ik} : R \to E, \forall A \in R, f_{ik} = \sum_{j=1}^{n} a_{ij} b_{kj},$$

then $O_{(n)}$ is the intersection of each set in the series

$$f_{ik}^{-1}(0), 1 \le i, k \le n, i \neq k, f_{ii}(1), 1 \le i \le n.$$

Since f_{ik} $(1 \le i, k \le n)$ is continuous mappings, each of the above sets is a closed set. So $O_{(n)}$ is a bounded closed set of R, which proves the compactness of $O_{(n)}$.

Because it is a compact Lie group in topological structure, it is called compact Lie group, so $O_{(n)}$ is compact Lie group.

Theorem 3. $O_{(n)}$ is unconnected.

In order to prove Theorem 3, the related definitions and lemmas are introduced.

Definition 5. Let A and B be two subsets of topological space X. If $(A \cap \overline{B}) \cup (B \cap \overline{A}) = \phi$, then the subsets A and B are isolated.

Definition 6. Let X be a topological space. If there are two non empty isolated subsets A and B in X, making $X = A \cup B$, then X is an unconnected space; otherwise, X is a connected space.

Lemma 2. If X is a topological space, the following conditions are equivalent:

(1) X is a disconnected space;

(2) There are two non empty closed subsets A and B in X, which make $A \cap B = \phi, A \cup B = X$ hold;

(3) There is an open and closed non empty true subset in X.

Now we prove Theorem 3.

Proof. Let $SO_{(n)}$ be the set of all orthogonal matrices whose determinant is 1, and S be the set of all orthogonal matrices whose determinant is -1.

Since $\det$: $SO_{(n)} \to E^1$ is a continuous mapping, and we know that the single point set $\{1\}$ is a closed set of E^1, $SO_{(n)} = \det(1)$. Under continuous mapping, the primitive image of any closed set is also a closed set, so $SO_{(n)}$ is also a closed set, and $SO_{(n)}$ is a closed set of $O_{(n)}$.

Similarly, we can prove that S is a closed set. Because $SO_{(n)} \cup S = O_{(n)}, SO_{(n)} \cap S = \phi$, and $SO_{(n)}$ and S are closed sets, it can be seen from lemma 2 that $O_{(n)}$ is unconnected.

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REFERENCES

[1] Y.F.Wang, S.M. Shi. Higher Algebra (Fourth Edition). Higher Education Press, 2013.
[2] F.J. Si. Properties and determination of real orthogonal matrix. Science and technology horizon, 2013 (17): 84-85.
[3] J.C. Xiong. Lecture notes of point collection topology. Higher education press, 1985.
[4] Z.D. Yan, Y.C. Xu. Lie group and Lie algebra. Higher education press, 1985.
[5] S.B. Chen.. Rigid body motion base invariants of spatial curves. Journal of Wuhan University of science and technology, 2003,26 (4)
[6] Department of mathematics, East China Normal University. Mathematical analysis (second) (Fourth Edition). Higher Education Press, 2001.

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