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Properties and Applications of Orthogonal Transformation

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***Abstract*—In this paper, based on the introduction of the definition and properties of orthogonal transformation, the one-to-one correspondence between orthogonal matrix and orthogonal transformation is used to study the application of orthogonal transformation in multiple integral, first type surface integral, Taylor formula of multivariate function, etc., and the application of orthogonal transformation is extended to some extent.**

***Index Terms*—Orthogonal transformation, orthogonal matrix, multiple integral, surface integral, multivariate function.**

I. Introduction

Modern mathematics and its application play an important role in the development of science and technology. It needs to give mathematical solutions to some problems, but these problems can only be solved by turning them into algebraic problems, so the importance of algebraic methods has been paid more and more attention by many scholars. After the application of algebraic method, some problems will become clear and easy to understand at a glance, so it is easy to give solutions. Orthogonal transformation method is one of the methods commonly used in modern mathematics and its application. It is necessary for us to study it comprehensively and systematically.

Orthogonal transformation is a very important problem in algebra. In Euclidean space, orthogonal transformation also appears as a special linear transformation. Because of its characteristics in this aspect, orthogonal transformation has become an important research object in the course of Higher Algebra [1-5]. In addition, it is also widely used in other fields, such as its application in integral operation.

Based on the introduction of the definition and properties of orthogonal transformation, this paper studies the application of orthogonal transformation in multiple integral, first type surface integral and Taylor formula of multivariate function.

II. Definition and Properties of Orthogonal Transformation

A. Definition of orthogonal transformation

Definition 1 let A be a linear transformation on Euclidean space V, if it keeps the inner product of vector unchanged, that is, for any $\alpha,\beta\in V$, there is $(\mathrm{A}\alpha,\mathrm{A}\beta)=(\alpha,\beta)$, then A is called orthogonal transformation.

B. Properties of orthogonal transformation

Theorem 1 suppose that A is a linear transformation of n-dimensional Euclidean space V, then the following four propositions are equivalent to each other:

1) A is the orthogonal transformation.

2) A keeps the length of the vector constant, that is, for any $\alpha\in V$, $|\mathrm{A}\alpha|=|\alpha|$ holds.

3) If $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ is a standard orthogonal basis, then $\mathrm{A}\varepsilon_1,\mathrm{A}\varepsilon_2,\cdots,\mathrm{A}\varepsilon_n$ is also a standard orthogonal basis.

4) The matrix of linear transformation A under any group of standard orthogonal basis is orthogonal matrix.

Property 1 The orthogonal transformation is a reversible linear transformation.

Property 2 let A , B be orthogonal transformation, then AB and A^{-1} are also orthogonal transformation.

III. Application of Orthogonal Transformation

A. Application of orthogonal transformation in multiple integral

In the calculation of multiple integrals, the most commonly used method to simplify the integrand is variable substitution. But variable substitution is very random. It needs to consider not only the expression of the integrand itself, but also the integral region and so on, so it is very troublesome to integrate with this method. In some special cases, we use orthogonal transformation is a very useful method.

Theorem 2 Let A be an orthogonal matrix, and its determinant is 1. Point $P(x,y,z)^T$ is transformed into $Q=(u,v,w)^T$ by orthogonal transformation $Q=AP$, region V_P in the original coordinate system is correspondingly transformed into surface V_Q in the new coordinate system, then

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$$\iiint_P f(P)dxdydz = \iiint_Q f(A^{-1}Q)dudvdw.$$

Proof. From $Q = AP$, we have $P = A^{-1}Q = A'Q$. Since Jacobian determinant

$$|J| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = |\det A'| = 1,$$

so we get

$$\iiint_P f(P)dxdydz = \iiint_Q f(A^{-1}Q)dudvdw. \qquad (1)$$

Example 1 For the continuous function $f(t)$ on $(-\infty,+\infty)$,

$$\iiint_{x^2+y^2+z^2\le 1} f(ax+by+cz)dxdydz = \pi\int_{-1}^{1}(1-u^2)f(ku)du$$

holds, where $k=\sqrt{a^2+b^2+c^2}$.

Proof. If $a=b=c=0$, then

$$\iiint_{x^2+y^2+z^2\le 1} f(ax+by+cz)dxdydz$$

$$= \iiint_{x^2+y^2+z^2\le 1} dxdydz = \frac{4}{3}\pi f(0),$$

i.e., the conclusion obviously holds. Set $k \ne 0$ below.

We extend the unit vector $\left(\frac{a}{k},\frac{b}{k},\frac{c}{k}\right)$ to the following orthogonal matrix

$$A=\begin{pmatrix} \frac{a}{k} & \frac{b}{k} & \frac{c}{k} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

We make the transformation

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det A' = \pm 1,$$

and $\Omega: x^2+y^2+z^2 \le 1$ becomes $\Omega': u^2+v^2+w^2 \le 1$. So we have

$$\iiint_\Omega f(ax+by+cz)dxdydz = \iiint_{\Omega'} f(ku)dudvdw$$

$$= \int_{-1}^{1} du \iint_{v^2+w^2\le 1-u^2} f(ku)dvdw$$

$$= \int_{-1}^{1} du\int_0^{2\pi} d\theta\int_0^{\sqrt{1-u^2}} \rho f(ku)d\rho$$

$$= \pi\int_{-1}^{1}(1-u^2)f(ku)du.$$

Therefore, when we study the problems related to multiple integration, if we encounter a very complex multiple integration, we should first think of using orthogonal transformation to reduce it to a multiple integration that is easy to solve, and then we can answer it.

B. The application of orthogonal transformation in the area division of the first type curve

If the orthogonal transformation is introduced into the first type of surface partition, its form will remain unchanged. Therefore, the orthogonal transformation can also be applied to the area division of the first type curve.

Let S: $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ be a smooth surface, and under the orthogonal transformation

$$X_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = AX = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

S becomes S', then for the continuous function $f(x,y,z)$ on S, it holds that

$$\iint_S f(X)dS = \iint_{S'} f(A'X)dS'. \qquad (2)$$

Example 2 Prove that

$$\iint_S f(ax+by+cz)dS = 2\pi\int_{-1}^{1} f[(a^2+b^2+c^2)^{\frac{1}{2}}u]du,$$

where S is the unit sphere $x^2+y^2+z^2=1$.

Proof. If $a=b=c=0$, the conclusion is obvious. Otherwise, let $(a^2+b^2+c^2)^{\frac{1}{2}}$, expand the unit vector $(\frac{a}{k},\frac{b}{k},\frac{c}{k})$ into a third-order orthogonal matrix, and make the orthogonal transformation

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then from formula (2), we get

$$\iint_{x^2+y^2+z^2=1} f(ax+by+cz)dS = \iint_{u^2+v^2+w^2=1} f(ku)dS'.$$

Thus, we have

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$$w^2 = 1 - u^2 - v^2, (u,v) \in D,$$

$$\frac{\partial w}{\partial u} = -\frac{u}{w}, \frac{\partial w}{\partial v} = -\frac{v}{w},$$

$$\sqrt{1+\left(\frac{\partial w}{\partial u}\right)^2+\left(\frac{\partial w}{\partial v}\right)^2} = \sqrt{1+\left(-\frac{u}{w}\right)^2+\left(-\frac{v}{w}\right)^2}$$

$$= \frac{1}{\sqrt{1-u^2-v^2}},$$

$$dS = \frac{1}{\sqrt{1-u^2-v^2}} dudv$$

$$\iint_{u^2+v^2+w^2=1} f(ku)dS = \iint_D f(ku)\frac{1}{\sqrt{1-u^2-v^2}} dudv.$$

Let $u = u$, $v = \sqrt{1-u^2}\sin\theta$, where $-1 \le u \le 1$, $0 \le \theta \le 2\pi$, then

$$\iint_D f(ku)\frac{1}{\sqrt{1-u^2-v^2}} dudv$$

$$= \int_{-1}^{1} f(ku)du \int_0^{2\pi} \frac{\sqrt{1-u^2}\cos\theta}{\sqrt{1-u^2}\cos\theta} d\theta$$

$$= 2\pi \int_{-1}^{1} f(ku)du,$$

that is,

$$\iint_S f(ax+by+cz)dS = 2\pi\int_{-1}^{1} f[(a^2+b^2+c^2)^{\frac{1}{2}}]du.$$

The proof is complete.

The above is the application of orthogonal transformation in integral calculation. In addition, orthogonal transformation is also widely used in other fields of mathematics.

C. Application of orthogonal transformation in Taylor formula of multivariate function

It is well known that, to find the Taylor formula of multivariate function $f(x_1, x_2, \cdots, x_n)$ in the neighborhood of a certain point, the biggest difficulty for us is to find the mixed partial derivative of the known function. But if we use the method of orthogonal transformation, we can easily get the mixed partial derivative, or even get the Taylor formula without the mixed partial derivative.

In mathematical analysis, Taylor's formula of binary function is defined by Taylor's theorem. Then we can get Taylor's formula of multivariate function through Taylor's theorem.

Theorem 3 (Taylor's theorem) If function $f(x_1, x_2, \cdots, x_n)$ has a continuous partial derivative up to $N+1$ on a neighborhood $\mathrm{U}(P_0)$ of point $P_0(x_1^0, x_2^0, \cdots, x_n^0)$, then for any point $(x_1^0+h_1, x_2^0+h_2, \cdots, x_n^0+h_n)$ in $\mathrm{U}(P_0)$, there is

$$f(x_1^0+h_1, x_2^0+h_2, \cdots, x_n^0+h_n) = f(x_1^0, x_2^0, \cdots, x_n^0) + \left(\frac{\partial}{\partial x_1}h_1 + \frac{\partial}{\partial x_2}h_2 + \cdots + \frac{\partial}{\partial x_n}h_n\right) f(x_1^0, x_2^0, \cdots, x_n^0)$$

$$+ \cdots + \frac{1}{n!}\left(\frac{\partial}{\partial x_1}h_1 + \frac{\partial}{\partial x_2}h_2 + \cdots + \frac{\partial}{\partial x_n}h_n\right)^n f(x_1^0, x_2^0, \cdots, x_n^0)$$

$$+ \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x_1}h_1 + \frac{\partial}{\partial x_2}h_2 + \cdots + \frac{\partial}{\partial x_n}h_n\right)^{n+1} f(x_1^0+\theta h_1, \cdots, x_n^0+\theta h_n), \tag{3}$$

where $0 < \theta < 1$.

We call formula (3) Taylor's formula of multivariate function.

Next we introduce orthogonal transformation.

If $A = (a_{ij})_{n\times n}$ is an orthogonal matrix, then $AA' = E, |A| = 1$ (right-handed).

Let $x = (x_1, x_2, \cdots, x_n)^T, y = (y_1, y_2, \cdots, y_n)^T$, then $x = A'y$ can be obtained by orthogonal transformation $y = Ax$. Then transpose to get the Taylor formula of $(x_1, x_2, \cdots, x_n)$ after orthogonal transformation in the neighborhood of a point.

In addition, in order to use orthogonal transformation more safely, the following two important theorems will be used.

Theorem 4 If there is $f(x) = f(A'y)$ under the orthogonal transformation $y = Ax$, then the value of function $f(x)$ at point $P_0(x_1^0, x_2^0, \cdots, x_n^0)$ is equal to the value of $f(A'y)$ at point $w_0 = (y_1^0, y_2^0, \cdots, y_n^0)$, where w_0 is uniquely determined by the value of $f(x)$ at point P_0 in the equation corresponding to the transformation $y = Ax$.

Theorem 5 If function $f(x_1, x_2, \cdots, x_n)$ has continuous partial derivatives of order $n+1$ on a neighborhood $\mathrm{U}(P_0)$ of point $P_0(x_1^0, x_2^0, \cdots, x_n^0)$, then through orthogonal transformation, $f(A'y)$ also has a continuous partial derivative of order $n+1$ in the neighborhood $\mathrm{U}(w_0)$ of point w_0, where $\mathrm{U}(w_0)$ is the neighborhood corresponding to $\mathrm{U}(P_0)$ under the transformation $y = Ax$.

The above two theorems show that we can safely introduce orthogonal transformation to solve Taylor formula of

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multivariate function. In order to get the transformed Taylor formula, only its inverse transformation is needed to return to the original variable.

Example 3. Find the Taylor formula of function $f(x,y,z)=\sin(x+y+z)^2$ at point $(0,0,0)$.

Solution: we know that the normal vector of $x+y+z=0$ is $(1,1,1)$, and its unit vector is $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$. Take this direction as the transformed u axis, and then take two axes v,w, so that they are orthogonal. For example, the following two axes

$$v=\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right),w=\left(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}}\right)$$

can be taken.

The above three vectors can form the orthogonal matrix

$$A=\begin{pmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\end{pmatrix}.$$

If we make a orthogonal transformation

$$(u,v,w)^T=A(x,y,z)^T,$$

then when $(x,y,z)=(0,0,0)$, $(u,v,w)=(0,0,0)$. Since $(x,y,z)=A'(u,v,w)^T$, so we have

$$x+y+z=\sqrt{3}u.$$

Therefore, we change the Taylor formula of $\sin(x+y+z)^2$ at the point $(0,0,0)$ into that of $\sin(3u^2)$ at the point $(0,0,0)$, that is, when $u=0$, the Taylor formula. This is a one variable function problem, and there are ready-made formulas to apply. Therefore ,we have

$$\sin(3u^2)=3u^2-\frac{(3u^2)^3}{3!}+\frac{(3u^2)^5}{5!}-\cdots+(-1)^{n-1}\frac{(3u^2)^{2n-1}}{(2n-1)!}$$
$$+(-1)^n\frac{\cos 3\theta u^2}{(2n+1)!}(3u^2)^{2n+1},0<\theta<1.$$

Since $u=\frac{x}{\sqrt{3}}+\frac{y}{\sqrt{3}}+\frac{z}{\sqrt{3}}$, we can get

$$\sin(x+y+z)^2=(x+y+z)^2-\frac{(x+y+z)^6}{3!}$$
$$+\frac{(x+y+z)^{10}}{5!}-\cdots+(-1)^{n-1}\frac{(x+y+z)^{4n-2}}{(2n-1)!}$$
$$+(-1)^n\frac{\cos[\theta(x+y+z)^2]}{(2n+1)!}(x+y+z)^{4n+2},0<\theta<1.$$

If the method of orthogonal transformation is also used in approximate calculation and extremum calculation, the transformed variable need not be changed back to the original variable. Therefore, orthogonal transformation can also be applied to the calculation of some mathematical models.

IV. Conclusion

In this paper, the definition of orthogonal transformation and an important theorem are given, and its proof is given. In addition, the application of orthogonal transformation in multiple integral, first type surface integral and Taylor formula of multivariate function is discussed, and different methods of using orthogonal transformation are given for different applications. Finally, the orthogonal transformation is extended to a more general form.

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