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Research on the application of orthogonal transformation in double integral and conditional extremum

Liang Fang[a,*], Lin Pang[b]

[a] College of Mathematics and Statistics, Taishan University, 271000, Tai'an, Shandong, China
[b] The circulation department of Library, Taishan University, 271000, Tai'an, Shandong, China
[*] Email:fangliang3@163.com

***Abstract*—Orthogonal transformation is an important kind of transformation in euclidean space. As a kind of special and commonly used transformation, it is often used in solving problems, especially in modern mathematics, and it plays a very important role in higher algebra. Therefore, the study of orthogonal transformation is of great significance. This paper mainly discusses the application of orthogonal transformation in double integral and conditional extremum.**

***Index Terms*—Orthogonal transformation; double integral; conditional extremum; orthogonal matrix; Euclidean Space.**

I. INTRODUCTION

With the continuous development and progress of mathematics, the mutual infiltration and influence among various fields of mathematics has become more and more important, especially the use of algebraic methods has become more prominent, and orthogonal transformation is one of them, it is a kind of important transformation in euclidean space. As a special and commonly used Matrix, it is often used in solving problems, especially in modern mathematics, and plays a very important role [1-6]. Orthogonal transformation is used in many problems of Algebra and geometry, and because of its different applications, it can be divided into rotation transformation, reflection transformation, slip reflection transformation and translation transformation. Nowadays, more and more people pay attention to the application of orthogonal transformation, especially in the fields of image processing, mathematics, physics, computer and Algorithm design. The application of this method will make the process from complex to simple, easy to understand and solve.

Orthogonal transformation is one of the most important transformations in euclidean space. This paper focuses on the application of orthogonal transformation in the double integral and conditional extremum.

II. APPLICATION OF ORTHOGONAL TRANSFORMATION IN MULTIPLE INTEGRALS

Multiple integrals are one of the main research objects in mathematical analysis. As an important mathematical tool, they are often used to study geometric problems of three-dimensional space, but they often encounter technical difficulties in calculating multiple integrals, just like the problem of double integral, most of them will use the method of variable substitution, but this method has the disadvantages of great difficulty and randomness, because it needs to consider not only the integral region of the integrand, but also need to consider its convenience. Therefore, in some cases, using orthogonal transformation is an effective method of variable substitution, which will be more convenient and effective.

Example 1. Prove that

$$\iint_s f(ax+by+c)dxdy = 2\int_{-1}^{1}\sqrt{1-u^2}\, f\left(u\sqrt{a^2+b^2}+c\right)du,$$

where S： $x^2+y^2\le 1$， $a^2+b^2\ne 0$ [5].

Proof. make orthogonal transformation

$$u=\frac{1}{\sqrt{a^2+b^2}}(ax,by),\quad v=\frac{1}{\sqrt{a^2+b^2}}(ay,bx),$$

then we have $x^2+y^2=u^2+v^2$. So $x^2+y^2\le 1$ can be transformed into $u^2+v^2\le 1$， and

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{a^2+b^2}\begin{vmatrix} a & -b \\ b & a \end{vmatrix}=1,$$

thus

$$\iint_s f(ax+by+c)dxdy = \iint_{u^2+v^2\le 1} f\left(\sqrt{a^2+b^2}u+c\right)dxdv.$$

and since

$$\left\{u^2+v^2\le 1\right\}=\left\{(u,v)\middle| -1\le u\le 1, -\sqrt{1-u^2}\le v\le\sqrt{1-u^2}\right\},$$

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so

$$\iint_{s} f(ax+by+c)\,dxdy$$
$$=\int_{-1}^{1}du\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f\left(u\sqrt{a^2+b^2}+c\right)dv$$
$$=\int_{-1}^{1} f\left(u\sqrt{a^2+b^2}+c\right)du\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}dv$$
$$=2\int_{-1}^{1}\sqrt{1-u^2}\,f\left(u\sqrt{a^2+b^2}+c\right)du.$$

Example 2. Given that $A=\left(a_{ij}\right)_{3\times 3}$ is a symmetric positive-definite matrix, find the volume bounded by an ellipsoid $\sum_{i,j=1}^{3} a_{ij}x_i x_j \le 1$.

Solution. Since $A=\left(a_{ij}\right)_{3\times 3}$ is a symmetric positive-definite matrix, so $\sum_{i,j=1}^{3} a_{ij}x_i x_j$ is a positive definite quadratic form.

According to the knowledge of higher algebra, there exists an orthogonal matrix T such that

$$T'AT=\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix},$$

where λ_1, λ_2, λ_3 are the three positive eigenvalues of matrix A.

Make the orthogonal transformation

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=T\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix},$$

and the transformation

$$\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{\lambda_1}} & 0 & 0\\ 0 & \frac{1}{\sqrt{\lambda_2}} & 0\\ 0 & 0 & \frac{1}{\sqrt{\lambda_3}}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}=U\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix},$$

Then $U=U'$ and $U'T'ATU=E$ is a third order identity matrix,

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=TU\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix},$$

$$\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}=\frac{\partial(x_1,x_2,x_3)}{\partial(y_1,y_2,y_3)}\times\frac{\partial(y_1,y_2,y_3)}{\partial(u_1,u_2,u_3)}=|T|\times|U|=\frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}},$$

$$\sum_{i,j=1}^{3}a_{ij}x_ix_j=(x_1,x_2,x_3)A\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=(u_1,u_2,u_3)U'T'ATU\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}$$
$$=(u_1,u_2,u_3)E\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}=u_1^2+u_2^2+u_3^2,$$

Therefore, we have

$$V=\iiint_{\sum_{i,j=1}^{3}a_{ij}x_ix_j\le 1} dx_1dx_2dx_3$$
$$=\iiint_{u_1^2+u_2^2+u_3^2\le 1}\left|\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\right|du_1du_2du_3$$
$$=\frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\iiint_{u_1^2+u_2^2+u_3^2\le 1}du_1du_2du_3$$
$$=\frac{4\pi}{3\sqrt{\lambda_1\lambda_2\lambda_3}}.$$

Example 3. Σ is a unit sphere $x^2+y^2+z^2=1$, prove that:

$$I=\iint_{\Sigma} f(ax+by+cz)ds=2\pi\int_{-1}^{1} f\left(\sqrt{a^2+b^2+c^2}\,u\right)du.$$

Proof. Make the orthogonal transformation

$$x=u,\ y=\sqrt{1-u^2}\cos v,\ z=\sqrt{1-u^2}\sin v,$$

where $D:-1\le u\le 1, 0\le v\le 2\pi$, then we have

$$\frac{\partial(y,z)}{\partial(u,v)}=\begin{vmatrix}\frac{-u}{\sqrt{1-u^2}}\cos v & -\sqrt{1-u^2}\sin v\\ \frac{u}{\sqrt{1-u^2}}\sin v & \sqrt{1-u^2}\cos v\end{vmatrix}=-u,$$

$$\frac{\partial(z,x)}{\partial(u,v)}=\sqrt{1-u^2}\cos v,$$

$$\frac{\partial(x,y)}{\partial(u,v)}=-\sqrt{1-u^2}\sin v.$$

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So, we get

$$p(u,v)=\sqrt{u^2+(1-u^2)(\cos^2 v+\sin^2 v)}$$
$$=\sqrt{u^2+1-u^2}=1,$$

thus $ds=p(u,v)dudv$, where D is on the plane VW.

The curved area is divided into double integral

$$\iint_D f\left(\sqrt{a^2+b^2+c^2}u\right)dudv=2\pi\int_{-1}^{1} f\left(\sqrt{a^2+b^2+c^2}u\right)du,$$

thus, we have

$$\iint_D f\left(\sqrt{a^2+b^2+c^2}u\right)dudv=\iint_\Sigma f(ax+by+cz)ds.$$

Example 4. Calculate the following triple integral

$$I=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(5x^2+6y^2+4z^2-4xy-4xz)}dxdydz.$$

Solution. Let

$$f(x,y,z)=5x^2+6y^2+4z^2-4xy-4xz,$$

whose corresponding matrix is

$$\begin{pmatrix} 5 & -2 & -2 \\ -2 & 6 & 0 \\ -2 & 0 & 4 \end{pmatrix}.$$

It is easy to know that it is a positive definite matrix.

Let its eigenvalue be λ_1, λ_2, λ_3, then $\lambda_1>0$, $\lambda_2>0$, $\lambda_3>0$ and $\lambda_1\lambda_2\lambda_3=|A|=80>0$.

Make the orthogonal transformation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

then we have $f(x,y,z)=\lambda_1u^2+\lambda_2v^2+\lambda_3w^2$, According to the property of orthogonal transformation, we get

$$I=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(5x^2+6y^2+4z^2-4xy-4xz)}dxdydz$$
$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(\lambda_1u^2+\lambda_2v^2+\lambda_3w^2)}dudvdw$$
$$=\int_{-\infty}^{+\infty} e^{-\lambda_1u^2}du\int_{-\infty}^{+\infty} e^{-\lambda_2v^2}dv\int_{-\infty}^{+\infty} e^{-\lambda_3w^2}dw$$
$$=\frac{\sqrt{\pi}}{\sqrt{\lambda_1}}\times\frac{\sqrt{\pi}}{\sqrt{\lambda_2}}\times\frac{\sqrt{\pi}}{\sqrt{\lambda_3}}$$
$$=\sqrt{\frac{\pi^3}{80}}.$$

III. APPLICATION OF ORTHOGONAL TRANSFORMATION IN CONDITIONAL EXTREMUM

Conditional extremum is a common problem in mathematics. There are many methods to solve this kind of problem. In this section, orthogonal transformation is applied to solve this kind of problem, which is a good method.

Example 5. Find the maximum and minimum values of $F(x,y)=x^2+y^2$ under the condition $3x^2+4xy+3y^2=1$.

Solution. $3x^2+4xy+3y^2=1$ and $F(x,y)=x^2+y^2$ are expressed as

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=1,$$

$$F(x,y)=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

The characteristic roots of $A=\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are 1 and 5.

Take $T=\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, let $\begin{pmatrix} x \\ y \end{pmatrix}=T\begin{pmatrix} X \\ Y \end{pmatrix}$, then

$$T^{-1}AT=T'AT=\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}.$$

On the other hand, since

$$\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix}=X^2+5Y^2,$$

$$F(x,y)=\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} X & Y \end{pmatrix}T'T\begin{pmatrix} X \\ Y \end{pmatrix}=X^2+Y^2,$$

the problem is transformed into finding the maximum and minimum value of $F(x,y)=x^2+y^2$ under the condition $X^2+5Y^2=1$. It is easy to find that when $X=0$, the minimum value is $\frac{1}{5}$; when $\frac{1}{5}$, the maximum value is 1.

IV. CONCLUSION

Because orthogonal transformation is in line with the algebraic trend of mathematical development, it can play an extremely important role in the field of mathematics. It can solve many complex problems well and even simplify them, so it has a wide range of applications in many fields. This paper focuses on the application of orthogonal transformation in multiple integral and conditional extremum, which achieves the effectiveness of theoretical knowledge The purpose of using and fuzziness problems is to make up for the defects of the existing methods and perfect the preciseness and accuracy of mathematics itself

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Authors' biography with Photo

Liang Fang was born in December 1970 in Feixian County, Linyi City, Shandong province, China. He is a professor at Taishan University. He obtained his PhD from Shanghai Jiaotong University in June, 2010. His research interests are in the areas of cone optimizations, numerical analysis, and complementarity problems.