ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

IJARMATE

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 6, Issue 5, May 2020

Discussion on the application of the minimum polynomials of matrices

Liang Fang[a], Rui Chen[b]

College of Mathematics and Statistics, Taishan University, 271000, Tai'an, Shandong, China

[a] Email:fangliang3@163.com
[b] Email: chenruimengting@163.com

***Abstract*—Based on the theory of minimum polynomials, this paper discusses the application of the minimum polynomials of matrices in dimensionality, diagonalization of square matrices, invertibility of matrix polynomials, differential equations and matrix equations.**

***Index Terms*—Minimum polynomial, matrix polynomial, dimension, diagonalization, matrix equation.**

I. INTRODUCTION

In solving some algebraic problems, such as the solution of matrix equations, the inverse of matrix function, and the solution of differential equations, especially in solving the initial value problem of linear differential equations with constant coefficients, the minimum polynomials of matrices can simplify the operation and calculate the standard matrix quickly. At the same time, the minimum polynomial is also an important part of the polynomial theory. It has important applications in judging whether the matrix can be diagonalized, and whether the matrix is similar and the structure of linear transformation. In addition to the application of matrix theory, minimum polynomials are also widely used in linear control systems and other fields [1-6]. This paper mainly discusses the applications of matrix polynomials in dimensionality, diagonalization, reversibility, differential equations and matrix equations.

II. APPLICATIONS OF THE MINIMUM POLYNOMIAL OF MATRIX

The minimum polynomials of matrices have very important applications. Next, we will introduce the applications of the minimum polynomials of matrices in real life from the aspects of dimension of linear space, simplified calculation, singularity discrimination and similarity diagonalization judgment.

A. *Finding dimension with minimum polynomials*

The dimension of linear space composed of all the real coefficient matrix polynomials corresponding to matrix A can be easily solved by using the minimum polynomial of the matrix, and the minimum polynomials play an important role in it. In this way, the problem of solving the dimension of linear space is transformed into the problem of solving the degree of the minimum polynomials of matrices

Lemma 1 Let all the real coefficient matrix polynomials corresponding to A form a linear space W, then the dimension of W is equal to the degree of the minimum polynomials of A (see [3] and the references therein).

Example 1. Let

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega^2\end{pmatrix},$$

where

$$\omega=\frac{-1+\sqrt{3}i}{2}.$$

All real coefficient polynomials of matrix A form the linear space W over the complex field. Find the dimension of W.

Solution: The characteristic polynomial of matrix is

$$\begin{aligned} f(\lambda)&=|\lambda \boldsymbol{E}-\boldsymbol{A}|.\\ &=(\lambda-1)(\lambda-\omega)(\lambda-\omega^2)\end{aligned}$$

It is easy to get

$$\omega^2=\frac{-1-\sqrt{3}i}{2},\omega^3=1.$$

Through calculation, it is easy to see that the minimum polynomial of A is

$$m(\lambda)=(\lambda-1)(\lambda-\omega)(\lambda-\omega^2).$$

So, we obtain

$$\dim(\boldsymbol{W})=\partial(\boldsymbol{m}(\lambda))$$

B. *Simple calculation of polynomials of matrix*

Next, we simplify the calculation of the minimum polynomial of matrix A, and use the minimum polynomial to find the matrix polynomial f of matrix $f(A)$.

Example 2. The matrix

$$\boldsymbol{A}=\begin{pmatrix}2 & 0 & 0\\ 1 & 1 & 1\\ 1 & -1 & 3\end{pmatrix},$$

$$f(\lambda)=\lambda^6+2\lambda^5-\lambda^4+3\lambda^2-2\lambda+1.$$

All Rights Reserved © 2020 IJARMATE

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 6, Issue 5, May 2020

Find the minimum and matrix polynomials of matrix A.

Solution. It is easy to find that the characteristic polynomial of A is

$$|\lambda E - A| = (\lambda - 2)^3.$$

Therefore, a factor of $(\lambda-2)^3$ is the minimum polynomial of A. By calculation, we can easily get

$$\boldsymbol{A} - 2\boldsymbol{E} \neq 0,$$

$$(A-2E)^2 = 0.$$

So the minimum polynomial of matrix A is

$$m(\lambda) = (\lambda-2)^2.$$

According to Lemma 1, the dimension of linear space W is 2. So there is a polynomial

$$p(\lambda) = a_0 + a_1\lambda,$$

such that

$$f(A) = p(A)$$

with

$$\begin{cases} \boldsymbol{f}(2) = \boldsymbol{p}(2), \\ \boldsymbol{f}'(2) = \boldsymbol{p}'(2). \end{cases}$$

Therefore,

$$\begin{cases} \boldsymbol{f}(2) = \boldsymbol{a}_0 + 2\boldsymbol{a}_1, \\ \boldsymbol{f}'(2) = \boldsymbol{a}_1. \end{cases}$$

Thus,

$$\begin{cases} \boldsymbol{a}_1 = \boldsymbol{f}'(2), \\ \boldsymbol{a}_0 = \boldsymbol{f}(2) - 2\boldsymbol{f}'(2). \end{cases}$$

So we get

$$\boldsymbol{f}(\boldsymbol{A}) = [\boldsymbol{f}(2) - \boldsymbol{f}'(2)]\boldsymbol{E} + \boldsymbol{f}'(2)\boldsymbol{A}$$
$$= \begin{bmatrix} \boldsymbol{f}(2) & 0 & 0 \\ \boldsymbol{f}'(2) & \boldsymbol{f}(2)-\boldsymbol{f}'(2) & \boldsymbol{f}'(2) \\ \boldsymbol{f}'(2) & -\boldsymbol{f}'(2) & \boldsymbol{f}'(2)+\boldsymbol{f}'(2) \end{bmatrix}.$$

Since

$$f(2) = 121, \quad f'(2) = 330,$$

so we have

$$\boldsymbol{f}(\boldsymbol{A}) = \begin{bmatrix} 121 & 0 & 0 \\ 330 & -209 & 330 \\ 330 & -330 & 451 \end{bmatrix}.$$

C. The singularity of discriminant matrix polynomial f(A)

The minimum polynomial of matrix $\boldsymbol{A}$ plays an indelible role in judging the singularity of matrix polynomials. It can distinguish the singularity by the rank of matrix $\boldsymbol{A}$.

Lemma 2 [2] Let A be a matrix with order n, $f(\lambda)$ is a polynomial, and the degree of $f(\lambda)$ is greater than zero, and the minimum polynomial of the matrix is $m(\lambda)$. Then

(1) If $f(\lambda)$ is divided by $m(\lambda)$, then $f(A)$ is degenerate.

(2) If the greatest common factor of $f(\lambda)$ and $m(\lambda)$ is $d(\lambda)$, then $rank(f(A)) = rank(d(A))$.

(3) The necessary and sufficient condition for polynomial $f(A)$ to be nondegenerate is $(\boldsymbol{f}(\lambda), \boldsymbol{m}(\lambda)) = 1$.

Example 3. Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix},$$

$$\boldsymbol{f}(\lambda) = \lambda^6 + 2\lambda^5 - \lambda^4 + 3\lambda^2 - 2\lambda + 1,$$

$$\boldsymbol{g}(\lambda) = \lambda^4 - 2\lambda^3 + 2\lambda^2 - 3\lambda - 2.$$

Find the minimum polynomial of matrix A and the rank of $g(A)$, judge the singularity of $f(A)$ and $g(A)$, and prove it.

Solution: According to the conclusion of the previous example, the minimum polynomial of A is

$$m(\lambda) = (\lambda-2)^2.$$

According to the theory of matrix polynomial,

$$d_1(\lambda) = (f(\lambda), m(\lambda)) = 1$$

is the greatest common factor of $f(\lambda)$ and $m(\lambda)$.

The greatest common divisor of $g(\lambda)$ and $m(\lambda)$ is

$$\boldsymbol{d}_2(\lambda) = (\boldsymbol{g}(\lambda), \boldsymbol{f}(\lambda)) = \lambda - 2,$$

which can be obtained by rolling phase division. From Lemma 2, $g(\lambda)$ is a singular matrix, and

$$rank(g(A)) = rank(d_2(A)).$$

So we have

$$\boldsymbol{d}_2(\boldsymbol{A}) = \boldsymbol{A} - 2\boldsymbol{E}$$
$$= \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix},$$

and

$$rank(g(A)) = rank(d_2(A)) = 1.$$

D. Judge whether the square matrix can be diagonalized

Generally, the eigenvalues of a matrix are used to judge whether a matrix can be diagonalized. Next, we use the least

All Rights Reserved © 2020 IJARMATE

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 6, Issue 5, May 2020

polynomial of the matrix to judge whether the matrix can be diagonalized. This method can not only get the results quickly and effectively, but also further deepen the understanding and mastery of knowledge.

Example 4. Judge whether matrix A can be diagonalized in the following three cases:

(1) $\boldsymbol{A}^2=\boldsymbol{E}$;

(2) $\boldsymbol{A}^2=\boldsymbol{A}$;

(3) $\boldsymbol{A}^m=0,\boldsymbol{m}>1$.

Solution. (1) If $A^2=E$, then

$$f(x)=x^2-1=(x+1)(x-1)$$

is the zeroing polynomial of A and has no multiple roots. From the properties of minimum polynomials, A can be diagonalized.

(2) If $\boldsymbol{A}^2=\boldsymbol{A}$, then

$$f(x)=x^2-x=x(x-1).$$

From the properties of minimum polynomials, A can be diagonalized.

(3) If $\boldsymbol{A}^m=0,\boldsymbol{m}>1$, then $f(x)=x^m$ is a zeroing polynomial. But it has multiple roots, so A can't be diagonalized.

E. Solving differential equations

In linear algebra, the problem of solving linear equations is often involved. Using the theoretical knowledge of matrix minimum polynomials, differential equations can be solved effectively and simply

Example 5. Solve the following differential equations

$$\begin{cases}\dfrac{d\boldsymbol{X}}{dt}=\boldsymbol{AX},\\ \boldsymbol{X}(0)=(0,1,1)^T,\end{cases}$$

where

$$A=\begin{pmatrix}6 & 2 & -2\\ -2 & 2 & 2\\ 2 & 2 & 2\end{pmatrix},$$

$$X=(x_1,x_2,x_3)^T.$$

Solution. The minimum polynomial of A is

$$m_A(\lambda)=(\lambda-2)(\lambda-4).$$

Let

$$f(At)=a_0(t)E+a_1(t)A.$$

$a_0(t)$ and $a_1(t)$ satisfy

$$\begin{cases}e^{2t}=a_0+2a_1,\\ e^{4t}=a_0+4a_1,\end{cases}$$

then

$$\begin{cases}a_0=-e^{4t}+2e^{2t},\\ a_1=\dfrac{1}{2}\left(e^{4t}-2e^{2t}\right).\end{cases}$$

So, we have

$$\boldsymbol{e}^{\boldsymbol{A}t}=\boldsymbol{f}(\boldsymbol{A}t)=\boldsymbol{a}_0(t)\boldsymbol{E}+\boldsymbol{a}_1(t)\boldsymbol{A}$$
$$=\begin{bmatrix}2\boldsymbol{e}^{4t}-\boldsymbol{e}^{2t} & \boldsymbol{e}^{4t}-\boldsymbol{e}^{2t} & -\boldsymbol{e}^{4t}+\boldsymbol{e}^{2t}\\ -\boldsymbol{e}^{4t}+\boldsymbol{e}^{2t} & \boldsymbol{e}^{2t} & \boldsymbol{e}^{4t}-\boldsymbol{e}^{2t}\\ \boldsymbol{e}^{4t}-\boldsymbol{e}^{2t} & \boldsymbol{e}^{4t}-\boldsymbol{e}^{2t} & \boldsymbol{e}^{2t}\end{bmatrix}.$$

From the theory of the definite solution of the differential equations, we can get the solution of the equations is

$$\boldsymbol{X}=\boldsymbol{e}^{\boldsymbol{A}t}\boldsymbol{X}(0)=(0,\boldsymbol{e}^{4t},\boldsymbol{e}^{4t})^{\boldsymbol{T}}.$$

III. CONCLUSION

In this paper, we generalize the application of the minimum polynomials by using the related theory of the minimum polynomials, including: reducing the degree of polynomials and finding matrix polynomials by using the method of division with remainder; getting dimension and a set of bases of linear space more succinctly according to some theorems of the minimum polynomials; solving matrix equation; judging whether the matrix can be right or not by judging whether the minimum polynomials of the matrix have multiple roots The application of minimum polynomials in the above aspects is illustrated. The application of matrix minimum polynomials can also be extended to other fields, so the application of matrix minimum polynomials is worth further study.

IV. ACKNOWLEDGMENT

The work is supported by Tai'an Science and Technology Development Project: Research on case teaching of linear algebra combined with learners practical ability (2019ZC251) and Teaching Reform and Research Project of Taishan University in 2018: The Theory of Probability Combining Real Life Application and Computer Simulation and the Teaching Reform of Mathematical Statistics Course.

REFERENCES

[1] Y.F. Wang, S.M. Shi. Advanced *Algebra*. Higher Education Press, 2013.
[2] Z.X. Yang. *Advanced Algebra*. Higher Education Press, 2007.
[3] J. Yao, Y.C. Zhang. *Engineering Matrix Method*. National Defense Industry Press, 2007.
[4] X.Y. Ma, L. Fang. *Selected Lectures on Advanced Algebra* Tianjin Education Press, 2010.
[5] Z.P Yang. *On the minimum polynomial of permutation matrix*. Journal of Qufu Normal University, 15 (3):9-11,1989.
[6] J.H. Wu. *Solution and application of matrix minimum polynomial*. Journal of Hanshan Normal University, 12:31-36, 2010.

All Rights Reserved © 2020 IJARMATE

ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 6, Issue 5, May 2020

Authors' biography with photos

Liang Fang is a professor at Taishan University. He obtained his PhD from Shanghai Jiaotong University in June, 2010. His research interests are in the areas of cone optimization, and complementarity problems.

Rui Chen is a lecturer at Taishan University. She obtained her master's degree from Shandong University in December, 2009. Her research interests are in the areas of application of probability theory, and applied statistics in recent years.

All Rights Reserved © 2020 IJARMATE