

Applications of Determinant and Matrix in Elementary Mathematics

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Abstract—In the course of higher algebra, determinants and matrices are very basic and important contents. They are not only widely used in higher mathematics, but also play a connecting role in the field of elementary mathematics. It is particularly important to know how to apply determinants and matrices to elementary mathematics so as to apply them to practice. This paper briefly expounds the definition and fundamentality of determinants and matrices. The applications of determinants and matrices in algebra and geometry are introduced, including factorization, sequence, vector and so on. The calculation methods and applications of determinants and matrices can not only deepen our understanding of mathematics, but also be more conducive to mathematics teaching in elementary mathematics.

Index Terms—Determinant; Matrix; Factorization; Sequence; Vector.

I. INTRODUCTION

In the field of higher mathematics, determinants and matrices are indispensable tools. In order to apply the knowledge of determinants and matrices flexibly to elementary mathematics, this paper will integrate some theories and calculation methods of determinants and matrices into solving elementary mathematics problems in the following aspects. The application of determinants and matrices in higher algebra is not only limited to the application of determinants and matrices. It is not only the foundation, but also the key to the study of advanced mathematical knowledge. Its application has already gone beyond the scope of algebra and has become the research object of elementary mathematics. In addition, it is not difficult to find that determinants and matrices also show unique advantages in elementary mathematics, such as factorization of polynomials easily and conveniently. This paper will pass some relevant theorems, inferences and example tables. The general application of explicit determinant and matrix in elementary mathematics actively cultivates divergent thinking, treats mathematical problems with unique and innovative vision, and combines the theoretical knowledge learned with the reality of life.

II. DEFINITION AND PROPERTIES OF DETERMINANTS

A. Definition of Determinants

Definition of the second-order determinant: The expression

$a_{11}a_{22} - a_{12}a_{21}$ is called a second-order determinant which is determined by $a_{11}, a_{12}, a_{21}, a_{22}$. The second-order determinant is symbolically expressed as

$$a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Definition of the third-order determinant: The expression

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \cdots - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

is called a third-order determinant which is determined by $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$. The second-order determinant is symbolically expressed as

$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Definition of determinant of order n : The determinant

$$\text{of order } n \text{ is } \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \text{ which is equal to}$$

the algebraic sum of the product of all elements taken out in different rows and columns, where $j_1 j_2 \cdots j_n$ is an arrangement of numbers $1, 2, \cdots, n$. Each item $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ is symbolized according to the following rules: when $j_1 j_2 \cdots j_n$ is even, the item $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ has a positive sign, and when $j_1 j_2 \cdots j_n$ is odd, the item $a_{1j_1} a_{2j_2} \cdots a_{nj_n}$ has a negative sign. This definition can be written as

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} a_{1j_1} a_{2j_2} \cdots a_{nj_n},$$

where $\sum_{j_1 j_2 \cdots j_n}$ is the sum of all all n-order permutations [1].

B. Relevant Properties of Determinants

(1) The determinant is interchangeable, and the value of the determinant remains unchanged [2], i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}.$$

(2) A row (column) of a determinant multiplied by a number is equivalent to multiplying the determinant by this number [2], i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

(3) If a row (column) in a determinant is the sum of two sets of numbers, then the determinant is the sum of the two determinants, and the other elements of the two determinants remain unchanged except for this row (column) [2], i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 + c_1 & b_2 + c_2 & \cdots & b_n + c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

(4) If two rows (columns) of a determinant are identical, then the determinant is zero [2], i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

(5) If two rows (columns) are proportional in the determinant, then the determinant is zero, i.e.,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = 0.$$

(6) Add a multiple of one row (column) to another row (column), and the determinant remains unchanged,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} + ca_{k1} & a_{i2} + ca_{k2} & \cdots & a_{in} + ca_{kn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{k1} & ca_{k2} & \cdots & ca_{kn} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

(7) If the position of two rows (columns) in a determinant is

exchanged, the determinant is inversely numbered,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

III. DEFINITION OF MATRIX AND RELATED THEORY

A. Definition of Matrix

The number table $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ with

m rows and n columns arranged by the number of $a_{ij} (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ on the number field P is called the $m \times n$ matrix on the number field P .

B. Relevant Theory of Matrix

(1) The rank of matrix is equal to the rank of row vector group, and it is equal to the rank of column vector group, which is the highest order non-zero subformula of matrix. The rank of matrix A is written by $r(A)$ [2].

(2) The rank of a matrix M is equal to $r \Leftrightarrow M$ has a non-zero r -order subformula, and all $r+1$ -order subformulas are zero.

(i) $r(M) \geq r \Leftrightarrow M$ has a non-zero r -order subformula;

(ii) $r(M) \leq r \Leftrightarrow$ all $r+1$ -order subformulas of M are zero.

(3) Elementary transformations of matrices

The following three kinds of elementary transformations are called matrices:

(i) Exchange the positions of certain two rows (columns) in a matrix;

(ii) Multiply a row (column) by a nonzero number;

(iii) Multiply one row (column) by k and add it to another row (column).

The elementary transformation of matrix does not change the rank of matrix, i.e., rank is the invariant of elementary transformation of matrix [5].

(3) Condition of equivalence

The necessary and sufficient condition for the rank of n -order square matrix A is equal to n is that $|A| \neq 0$, which is equivalent to linear equations $AX = 0$ have only zero solutions, or the row (column)

vector groups of A is linearly independent.

The necessary and sufficient condition for the rank of n -order square matrix A is less than n is that $|A| = 0$, which is equivalent to linear equations $AX = 0$ have nonzero solutions, or the row (column) vector groups of A is linearly dependent [5].

IV. THE APPLICATION OF DETERMINANT AND MATRIX IN ELEMENTARY MATHEMATICS

A. Application of determinants in algebra

Solving equations is a basic problem in algebra, especially in algebras learned in middle school. Solving equations plays an important role. In algebras learnt in middle school, we have not only solved one-dimensional and binary first-order equations and equations, but also three-dimensional and quaternary first-order equations. Here we will give examples to illustrate determinants in solving linear equations.

(i) The Application of Determinants in Linear Equations

The related theory knowledge of linear equations is very important in the study of higher mathematics. Next, it will permeate the elementary mathematics through the application of determinants and matrices.

Example 1 Solve binary system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1, \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

Solution: According to Cramer's law, when

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0, \text{ the system of linear equations has a unique}$$

solution, that is

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}},$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

Example 2 The function

$$f(x) = ax^3 + bx^2 + cx + d$$

Satisfies

$$f(-1) = 0, f(1) = -6,$$

$$f(2) = -9, f(3) = -4.$$

Find the analytic formula of $f(x)$ [3].

Solution: From known conditions, we have

$$\begin{cases} a(-1)^3 + b(-1)^2 + c(-1) + d = 0, \\ a \cdot 1^3 + b \cdot 1^2 + c \cdot 1 + d = -6, \\ a \cdot 2^3 + b \cdot 2^2 + c \cdot 2 + d = -9, \\ a \cdot 3^3 + b \cdot 3^2 + c \cdot 3 + d = -4. \end{cases}$$

Consider the above formula as a system of linear equation about a, b, c, d . Its coefficient determinant is Vandermonde determinant

$$\begin{aligned} d &= \begin{vmatrix} (-1)^3 & (-1)^2 & -1 & 1 \\ 1^3 & 1^2 & 1 & 1 \\ 2^3 & 2^2 & 2 & 1 \\ 3^3 & 3^2 & 3 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 2 & 3 \\ (-1)^2 & 1^2 & 2^2 & 3^2 \\ (-1)^3 & 1^2 & 2^3 & 3^3 \end{vmatrix} = 48. \\ d_1 &= \begin{vmatrix} 0 & (-1)^2 & -1 & 1 \\ -6 & 1^2 & 1 & 1 \\ -9 & 2^2 & 2 & 1 \\ -4 & 3^2 & 3 & 1 \end{vmatrix} = 48, \\ d_2 &= \begin{vmatrix} (-1)^3 & 0 & -1 & 1 \\ 1^3 & -6 & 1 & 1 \\ 2^3 & -9 & 2 & 1 \\ 3^3 & -4 & 3 & 1 \end{vmatrix} = -96, \\ d_3 &= \begin{vmatrix} (-1)^3 & (-1)^2 & 0 & 1 \\ 1^3 & 1^2 & -6 & 1 \\ 2^3 & 2^2 & -9 & 1 \\ 3^3 & 3^2 & -4 & 1 \end{vmatrix} = -192, \\ d_4 &= \begin{vmatrix} (-1)^3 & (-1)^2 & -1 & 0 \\ 1^3 & 1^2 & 1 & -6 \\ 2^3 & 2^2 & 2 & -9 \\ 3^3 & 3^2 & 3 & -4 \end{vmatrix} = -48. \end{aligned}$$

So we have

$$\begin{aligned} a &= \frac{d_1}{d} = 1, \quad b = \frac{d_2}{d} = -2, \\ c &= \frac{d_3}{d} = -4, \quad d = \frac{d_4}{d} = -1. \end{aligned}$$

Return a, b, c, d to the original function and get

$$f(x) = x^3 - 2x^2 - 4x - 1.$$

(ii) Decomposition of Factor by Using Determinant

As one of the important tools for studying mathematics, determinant is very convenient and practical. It is an innovative method to solve the problem of decomposition factor by applying the related properties of determinant in the field of mathematics. It is beneficial to form divergent thinking. To use determinant to decompose factor, the key point is to write the given polynomial into determinant form, among which determinant is the determinant. In general, an algebraic expression can always be regarded as the result of the subtraction of two formulas, and each of them can be regarded as the result of the multiplication of two factors. That is to say, an algebraic expression A can be regarded as $a_{11}a_{22} - a_{12}a_{21}$. Thus, A can be written in the form of a second-order determinant

$$A = a_{11}a_{22} - a_{12}a_{21} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Some polynomials can be factorized by using (2) of the related properties of the determinants mentioned above.

Example 3 Factorization factor:

$$(cd - ab)^2 - 4bc(a - c)(b - d).$$

Solution:

$$\begin{aligned} &(cd - ab)^2 - 4bc(a - c)(b - d) \\ &= \begin{vmatrix} cd - ab & 2(ab - bc) \\ 2(bc - cd) & cd - ab \end{vmatrix} \\ &= \begin{vmatrix} cd - ab & ab + cd - 2bc \\ 2(bc - cd) & -(ab + cd - 2bc) \end{vmatrix} \\ &= (ab + cd - 2bc) \begin{vmatrix} cd - ab & 1 \\ 2(bc - cd) & -1 \end{vmatrix} \\ &= (ab + cd - 2bc)^2. \end{aligned}$$

Theorem 1 [4] assume that

$p(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$ is a univariate polynomial over the number field P , then

$$p(x) = \begin{vmatrix} x & -1 & 0 & \dots & 0 \\ 0 & x & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -1 \\ a_n & a_{n-1} & a_{n-2} & \dots & a_0x + a_1 \end{vmatrix}.$$

By using this theorem, an n -th polynomial can be written as a determinant of n -th order, and then the determinant can be simplified according to the nature of the determinant, making it the product of some irreducible factors.

Example 4 Decompose the following polynomial into factorials

$$5x^4 + 24x^3 - 15x^2 - 118x + 24.$$

Solution:

$$5x^4 + 24x^3 - 15x^2 - 118x + 24$$

$$\begin{aligned} &= \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ 24 & -118 & -15 & 5x+24 \end{vmatrix} \\ &= \begin{vmatrix} x & -1 & 0 & 0 \\ 0 & x & -1 & 0 \\ 0 & 0 & x & -1 \\ 24 & -118 & 5x^2+24x-15 & 0 \end{vmatrix} \\ &= \begin{vmatrix} x & -1 & 0 \\ 0 & x & -1 \\ 24 & -118 & 5x^2+24x-15 \end{vmatrix} \\ &= \begin{vmatrix} x & 5x-1 & 0 \\ 0 & x & -1 \\ 24 & 2 & 5x^2+24x-15 \end{vmatrix} \\ &= \begin{vmatrix} x & 5x-1 & 5(5x-1) \\ 0 & x & 5x-1 \\ 24 & 2 & 5x^2+24x-5 \end{vmatrix} \\ &= (5x-1) \begin{vmatrix} x & 5x-1 & 5 \\ 0 & x & 1 \\ 24 & 2 & x+5 \end{vmatrix} \\ &= (5x-1) \begin{vmatrix} x & x+3 \\ 8 & (x+3)(x+2) \end{vmatrix} \\ &= (5x-1)(x-2)(x+3)(x+4). \end{aligned}$$

(iii) The Application of Determinant in the Study of Sequence Problems

Theorem 2 [5] If three numbers x_1, x_2, x_3 are not equal, they are arithmetic progression, and

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

Then y_1, y_2, y_3 are also arithmetic progression.

Corollary 2.1 [5] a_m, a_n, a_k is the m -th term, n -th term and k -th term of the sequence of arithmetic progression, respectively, if and only if

$$\begin{vmatrix} m & a_m & 1 \\ n & a_n & 1 \\ k & a_k & 1 \end{vmatrix} = 0. \quad (2.1)$$

Proof: Sufficiency. We can know from the condition

$$\begin{vmatrix} m & a_m & 1 \\ n & a_n & 1 \\ k & a_k & 1 \end{vmatrix} = 0 \quad \text{that } (m, a_m), (n, a_n), (k, a_k) \text{ are}$$

collinear. We assume that the equation of the straight line is $y = ax + b$, then the general term formula of the sequence containing a_m, a_n, a_k is $a_n = an + b_n$.

Necessity. It is known that a_m, a_n, a_k are the m -th term, n -th term and k -th term of the sequence of arithmetic progression. If the first item of the sequence in which they are located is a_1 and the tolerance is d , then

$$\begin{aligned} \begin{vmatrix} m & a_m & 1 \\ n & a_n & 1 \\ k & a_k & 1 \end{vmatrix} &= \begin{vmatrix} m & a_1 + (m-1)d & 1 \\ n & a_1 + (n-1)d & 1 \\ k & a_1 + (k-1)d & 1 \end{vmatrix} \\ &= \begin{vmatrix} m-k & (m-k)d & 0 \\ n-k & (n-k)d & 0 \\ k & a_1 + (k-1)d & 1 \end{vmatrix} \\ &= \begin{vmatrix} m-k & (m-k)d \\ n-k & (n-k)d \end{vmatrix} = 0. \end{aligned}$$

Corollary 2.2 [5] If a, b, c become an arithmetic progression, and the tolerance is $d \neq 0$, then the necessary and sufficient condition for x, y, z to be a geometric progression is

$$I = \begin{vmatrix} a & \log_m x & 1 \\ b & \log_m y & 1 \\ c & \log_m z & 1 \end{vmatrix} = 0 \quad (m > 0, m \neq 1). \quad (2.2)$$

Proof: Sufficiency. Assume that x, y, z become a geometric progression, then $y^2 = xz$. Taking logarithms on both sides of (2.2) gives $2\log_m y = \log_m x + \log_m z$, that is, $\log_m x, \log_m y, \log_m z$ are arithmetic progression. From the theorem above, we have

$$I = \begin{vmatrix} a & \log_m x & 1 \\ b & \log_m y & 1 \\ c & \log_m z & 1 \end{vmatrix} = 0.$$

Necessity. Assume

$$I = \begin{vmatrix} a & \log_m x & 1 \\ b & \log_m y & 1 \\ c & \log_m z & 1 \end{vmatrix} = 0 \quad (m > 0, m \neq 1),$$

From the theorem above, we know that $\log_m x, \log_m y, \log_m z$ are arithmetic progression. Thus we have $2\log_m y = \log_m x + \log_m z$. Furthermore, we get $y^2 = xz$. Therefore, x, y, z are geometric progression.

Example 5 In $\triangle ABC$, $\tan A$ is the tolerance of the sequence of arithmetic progression with the third term -4 and the seventh term 4 . $\tan B$ is the common ratio of the sequence of geometric progression with the third term $\frac{1}{3}$ and the sixth term 9 . Prove that $\triangle ABC$ is an acute triangle [5].

Proof: From the conditions of the question, formula (2.1), (2.2) shows that the first term of the sequence of arithmetic progression is $-4 - 2\tan A$, and the first term of the sequence of geometric progression is $\frac{1}{3\tan^2 B}$. So, we have

$$\begin{vmatrix} 1 & -4 - 2\tan A & 1 \\ 3 & -4 & 1 \\ 7 & 4 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & \lg \frac{1}{3\tan^2 B} & 1 \\ 3 & \lg \frac{1}{3} & 1 \\ 6 & \lg 9 & 1 \end{vmatrix} = 0.$$

According to the related properties of determinants, the above two formulas are simplified, obtain $\tan A = 2$, $\tan B = 3$. So

$$\begin{aligned} \tan C &= \tan[\pi - (A + B)] \\ &= -\tan(A + B) = -\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1 \end{aligned}$$

Thus, $\angle C = 45^\circ$. Since $\tan A = 2 > 0$ and $\tan B = 3 > 0$, and $0^\circ < \angle A, \angle B < 180^\circ$, we get both A and B are acutes, so $\triangle ABC$ is an acute triangle.

B. Application of Determinants in Geometry

(i) Expressing the area of a triangle by determinant

Theorem 3 [6] The area of a triangle $\triangle ABC$ with three points $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ as its vertex in the plane is equal to

$$S_{\triangle ABC} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}. \quad (3.1)$$

Proof: The points $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ on the

plane are regarded as the points in three-dimensional space, whose coordinates are $(x_1, y_1, z), (x_2, y_2, z), (x_3, y_3, z)$, respectively, where k is an arbitrary constant.

Then we get

$$\overrightarrow{BA} = (x_2 - x_1, y_2 - y_1, 0),$$

$$\overrightarrow{CA} = (x_3 - x_1, y_3 - y_1, 0).$$

So

$$\begin{aligned} \overrightarrow{BA} \times \overrightarrow{CA} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \vec{k} \\ &= (0, 0, \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}) \end{aligned}$$

Therefore, the area of a triangle $\triangle ABC$ is:

$$\begin{aligned} S &= \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{CA}| \sin \langle \overrightarrow{BA}, \overrightarrow{CA} \rangle \\ &= \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{CA}| = \frac{1}{2} \sqrt{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix}^2} \\ &= \frac{1}{2} \left| \begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} \right| \\ &= \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \right| \\ &= \frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|. \end{aligned}$$

The determinant can be used not only to represent the area of a triangle, but also to solve the plane equation.

Theorem 4 [7] The equation of the plane α over three points $A(x_1, y_1, z_1), B(x_2, y_2, z_2), C(x_3, y_3, z_3)$ is as follows:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Theorem 5 [8] If the equation of plane α is $Ax + By + Cz + D = 0$, then the distance from point $P(x_0, y_0, z_0)$ outside the plane to the plane is

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Example 6 Given that the edge length of a square $ABCD$ is 4, $CG \perp ABCD$, $CG = 2$, E , F are the midpoint of AB and AD , respectively. Find the distance from the point B to the plane EFG .

Solution: For the purpose of this question, we establish the space rectangular coordinate system $C-xyz$ as shown in the following figure 1.

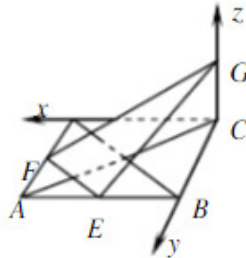


Figure 1

It is easy to see that The coordinates of B, E, F, G are $B(0,4,0), E(2,4,0), F(4,2,0), G(0,0,2)$, respectively.

Therefore, the equation of the plane EFG is :

$$\begin{vmatrix} x-0 & y-0 & z-2 \\ 2-0 & 4-0 & 0-2 \\ 4-0 & 2-0 & 0-2 \end{vmatrix} = 0.$$

After reorganizing, we have

$$-4x - 4y - 12z + 24 = 0,$$

i.e., $x + y + 3z - 6 = 0$.

Therefore, the distance from $B(0,4,0)$ to plane EFG is

$$d = \frac{|0+4+0-6|}{\sqrt{1+1+9}} = \frac{2\sqrt{11}}{11}.$$

(ii) Application of Determinant in Solving Vector Problem

Definition Let two non-collinear vectors

$$\vec{e}_1 = (x_1, y_1, z_1), \vec{e}_2 = (x_2, y_2, z_2)$$

in plane α , then determinant $\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$ is a normal

vector of plane α , which is recorded as \vec{n} [9].

Example 7 In right prism $ABC-A_1B_1C_1$, $AB = AC = \frac{1}{2}AA_1$, $\angle BAC = 90^\circ$. D is the midpoint of prism B_1B . Find a normal vector of plane ADC .

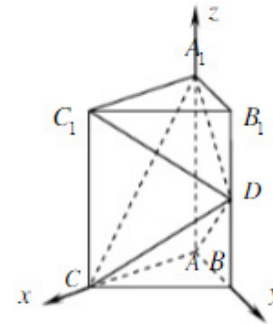


Figure 2

Solution: As shown in the figure, if the space rectangular coordinate system $A-xyz$ is established, then

$$A(0,0,0), A_1(0,0,2), B(0,1,0),$$

$$B_1(0,1,2), C(1,0,0), D(0,1,1).$$

Taking two non-collinear vectors $\vec{AD} = (0,1,1)$, $\vec{AC} = (1,0,0)$ in the plane ADC , we can get a normal vector of the plane as

$$\begin{aligned} \vec{n} = \vec{AD} \times \vec{AC} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} \\ &= \vec{j} - \vec{k} = (0,1,-1). \end{aligned}$$

V. CONCLUSIONS

Determinants and matrices are not only the basis of higher mathematics research, but also the extension and expansion of elementary mathematics research. In view of the research of determinants and matrices in higher mathematics, the fundamental purpose is to stand in the perspective of Higher Algebra and overlook the contents of the new curriculum standard of middle school mathematics. Flexible use of the study and research of higher algebra can make better progress between higher mathematics and elementary mathematics. In addition, in the face of the difficult problems in middle school mathematics, we can not only use the knowledge learned in middle school to solve these problems. Through the sublimation and improvement of knowledge in Higher Mathematics and the experience of some mathematical ideas and methods, we can effectively solve many problems in middle school. A mathematical problem that has not been explained clearly. This has certain practical significance for the use of modern mathematics ideas, principles and methods in middle school mathematics teaching.

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