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Application of Calculus Thought in Probability Theory and Mathematical Statistics

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***Abstract*—This article mainly introduces the application of Poisson integral, gamma function and micro increment method in the course of probability theory and mathematical statistics, and embodies the importance of calculus theory to probability theory and mathematical statistics. The application of calculus in the development of probability theory and mathematical statistics in the future is forecasted.**

***Key words*—Probability theory and mathematical statistics, calculus thinking, application.**

I. INTRODUCTION

A. Calculus Thought Introduction

Calculus is a branch of mathematics that studies the differentiation, integration, and related concepts and applications of functions. It is a basic discipline of mathematics. The main contents include limit, continuity, derivative, differential, indefinite integral, definite integral, partial derivative, full differential, double integral, curve integral, surface integral, etc.. The core idea is the limit, and the limit theory is reflected in almost all knowledge points of calculus [1].

B. The Development of Probability Theory and Mathematical Statistics

Probability theory and mathematical statistics is an important public basic course in colleges and universities, but it is different from other public basic courses in mathematics such as higher mathematics and linear algebra. It is a very practical mathematics discipline that studies the statistical regularity of random phenomena. Random phenomena exist in various fields and aspects of real life. Therefore, this discipline has a wide range of applications in many fields. At the same time, it is different from those professional courses that are directly related to engineering projects. The courses of probability theory and mathematical statistics belong to basic mathematics courses, passing on mathematics ideas and using mathematics as a tool to solve practical problems. Specifically, the course of probability theory and mathematical statistics teaches students the basic ideas and basic methods for dealing with random phenomena, and trains students to use the theory and methods of probability statistics to analyze and solve practical problems [2].

C. The Relationship between Calculus and Probability Theory and Mathematical Statistics

Calculus and probability theory and mathematical statistics are two very important mathematics disciplines in colleges and universities. They are compulsory courses for various majors in science and engineering in colleges and universities, and provide necessary mathematical tools for follow-up courses. Although the two development paths are far apart, there is a close relationship between the two. Calculus is the basis of probability theory and mathematical statistics. Probability theory and mathematical statistics are the continuation of calculus [3]. In university courses, calculus is also set up first, followed by probability theory and mathematical statistics. To reveal the penetration of calculus in probability theory and mathematical statistics, and to apply the ideas and methods of calculus cleverly to probability theory and mathematical statistics, is a problem that deserves our attention. The establishment of calculus has greatly promoted the development of mathematics. In the past, many problems in elementary mathematics were helpless. Using calculus, they often solved the problem and showed the extraordinary power of calculus. The development and progress of probability theory and mathematical statistics courses cannot be separated from the application of calculus thoughts. With the help of calculus, probability theory and mathematical statistics courses have achieved today's achievements.

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II. Application of Calculus Thought in Probability Theory and Mathematical Statistics

A. Application of Poisson integral in probability theory and mathematical statistics

Abnormal integral $\int_{-\infty}^{\infty} e^{-x^2}dx$ is called Poisson integrals, which have a good application in probability theory, we know that $\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}$ [4].

If the random variable obeys the normal distribution , that is $X \sim N\left(\mu,\sigma^2\right)$, It is easy to calculate its mathematical expectations and variance using Poisson integrals. The probability density function of the known normal distribution is

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}},x\in R.$$

By the definition of mathematical expectations, there is

$$E(X)=\int_{-\infty}^{\infty}x\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx,$$

Let $z=\frac{x-\mu}{\sigma}$, the above formula become

$$E(X)=\int_{-\infty}^{\infty}(\sigma z+\mu)\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(\sigma z)^2}{2\sigma^2}}d(\sigma z+\mu)$$

$$=\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}ze^{-\frac{z^2}{2}}dz+\frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}dz=\mu.$$

By the definition of variance, there is

$$D(X)=\int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx.$$

Let $z=\frac{x-\mu}{\sigma}$, the above formula become

$$D(X)=\int_{-\infty}^{\infty}(\sigma z)^2\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(\sigma z)^2}{2\sigma^2}}d(\sigma z+\mu)$$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^2e^{-\frac{z^2}{2}}dz$$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\left(-ze^{-\frac{z^2}{2}}\Big|_{-\infty}^{\infty}+\int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}dz\right)$$

$$=\frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{z^2}{2}}dz=\sigma^2.$$

If the random variable obeys the two-dimensional normal distribution, that is $(X,Y)\sim N\left(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho\right)$, the Poisson integral can be used to discuss its edge distribution. Because its joint probability density function is:

$$f(x,y)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{\frac{1}{2\left(1-\rho^2\right)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\},$$

$x\in R, y\in R$.

The edge probability density function for X can be solved as follows:

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{\frac{1}{2\left(1-\rho^2\right)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}dy$$

$$=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{(x-\mu_1)^2}{\sigma_1^2}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{y-\mu_2}{\sigma_2}-\rho\frac{x-\mu_1}{\sigma_1}\right)^2\right]dy$$

Let $t=\frac{1}{\sqrt{1-\rho^2}}\left(\frac{y-\mu_2}{\sigma_2}-\rho\frac{x-\mu_1}{\sigma_1}\right)$, and $dy=\sigma_2\sqrt{1-\rho^2}dt$, by Poisson integral

$$f_X(x)=\frac{1}{2\pi\sigma_1}e^{-\frac{(x-\mu_1)^2}{\sigma_1^2}}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}dt=\frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{(x-\mu_1)^2}{\sigma_1^2}},$$

$x\in R$.

In the same way,

$$f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_2}e^{-\frac{(y-\mu_2)^2}{\sigma_2^2}},\quad y\in R.$$

The Poisson integral is also applied in the logarithmic normal distribution. If the function of the random variable $Y=\ln X$ obeys the normal distribution $N\left(\mu,\sigma^2\right)$, X is said to obey the logarithmic normal distribution [5].

Assuming that the random variable X obeys the logarithmic normal distribution, the origin moment can be easily obtained using the Poisson integral. Let $Y=\ln X$, and $X=e^Y$,so

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$$E\left(X^k\right)=E\left(e^{kY}\right)$$

$$=\int_{-\infty}^{\infty}e^{ky}\cdot\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(y-\mu)^2}{2\sigma^2}}dy$$

$$=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{y^2-2\left(\mu+k\sigma^2\right)y+\mu^2}{2\sigma^2}}dy$$

$$=e^{k\mu+\frac{k^2\sigma^2}{2}}\cdot\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{\left(y-\mu-k\sigma^2\right)}{2\sigma^2}}dy.$$

Let $\frac{y-\mu-k\sigma^2}{\sigma}=t$, we have

$$E\left(X^k\right)=\frac{1}{\sqrt{2\pi}}e^{k\mu+\frac{k^2\sigma^2}{2}}\cdot\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}dt=e^{k\mu+\frac{k^2\sigma^2}{2}}.$$

The Poisson integral also has a good application in other distributions. Let the probability density of a two-dimensional random variable (X,Y) be:

$$f(x,y)=\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}(1+\sin x\sin y),x\in R,y\in R,$$

The Poisson integral can be used to calculate its edge probability density function [6].

Actually,

$$f_X(x)=\int_{-\infty}^{\infty}f(x,y)dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dy+\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}\sin x\sin y dy$$

$$=\frac{1}{2\pi}e^{-\frac{x^2}{2}},x\in R.$$

As well as being available,

$$f_Y(y)=\frac{1}{2\pi}e^{-\frac{y^2}{2}},y\in R.$$

Poisson integral is an abnormal integral in calculus. Its application in probability theory fully reflects the organic combination of calculus theory and probability theory.

B. Application of Gamma function in Probability Theory and Mathematical Statistics

The variable-containing integral $\Gamma(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-t}dt$ is called a gamma function. It is an important function in calculus. Its domain is the interval $(0,+\infty)$, which has a good application in the exponential distribution, normal distribution, convolution calculation in probability theory [7].

1) Application of Gamma Function in the Calculation of Digital Characteristics of Index Distribution

Let the random variable X obey the exponential distribution with a parameter of λ, and its probability density is:

$$f(x)=\begin{cases}\lambda e^{-\lambda x},x\geq 0\\ 0,x<0\end{cases},（\lambda>0）.$$

I. X 's mathematical expectations

$$E(X)=\int_{-\infty}^{+\infty}xf(x)dx=\int_0^{+\infty}x\cdot\lambda e^{-\lambda x}dx.$$

Make a variable substitution $\lambda x=t$, the special form of the gamma function is obtained, namely:

$$E(X)=\frac{1}{\lambda}\int_0^{+\infty}te^{-t}dt=\frac{\Gamma(2)}{\lambda}=\frac{1}{\lambda}.$$

II. X 's variance

$$D(X)=E\left(X^2\right)-E^2(X)=\int_0^{+\infty}x^2\cdot\lambda e^{-\lambda x}dx-\frac{1}{\lambda^2}.$$

Make a variable substitution $\lambda x=t$, the special form of the gamma function is obtained, namely:

$$D(X)=\frac{1}{\lambda^2}\int_0^{+\infty}t^2e^{-t}dt-\frac{1}{\lambda^2}=\frac{\Gamma(3)}{\lambda^2}-\frac{1}{\lambda^2}=\frac{2!}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}.$$

III. X 's origin moment

$$\mu_k=E\left(X^k\right)=\int_0^{+\infty}x^k\cdot\lambda e^{-\lambda x}dx,$$

Make a variable substitution $\lambda x=t$, the special form of the gamma function is obtained, namely:

$$\mu_k=\frac{1}{\lambda^k}\int_0^{+\infty}t^ke^{-t}dt=\frac{\Gamma(k+1)}{\lambda^k}=\frac{k!}{\lambda^k},k=1,2,3,\cdots.$$

IV. X 's central moment

$$E\left\{\left[X-E(X)\right]^k\right\}=\int_0^{+\infty}\left(x-\frac{1}{\lambda}\right)^k\cdot\lambda e^{-\lambda x}dx$$

$$=\frac{1}{\lambda^{k-1}}\int_0^{+\infty}(\lambda x-1)^ke^{-\lambda x}dx$$

$$=\frac{1}{\lambda^{k-1}}\int_0^{+\infty}\left[C_k^0(\lambda x)^k-C_k^1(\lambda x)^{k-1}+\cdots+(-1)^kC_k^k(\lambda x)^0\right]e^{-\lambda x}dx,$$

Make a variable substitution $\lambda x=t$, the special form of the gamma function is obtained, namely:

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$$E\left\{\left[X-E(X)\right]^k\right\}$$
$$=\frac{C_k^0\Gamma(k+1)}{\lambda^k}-\frac{C_k^1\Gamma(k)}{\lambda^k}+\cdots+(-1)^k\frac{C_k^k}{\lambda^k},k=1,2,3\cdots.$$

2) Application of Gamma Function in the Central Distance Calculation of Normal Distribution

If the random variable obeys the normal distribution, that is, $X \sim N(\mu,\sigma^2)$,

$$E\left\{\left[X-E(X)\right]^k\right\}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^k e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx.$$

Make a variable substitution $\frac{x-\mu}{\sigma}=t$, we have

$$E\left\{\left[X-E(X)\right]^k\right\}=\frac{\sigma^k}{\sqrt{2\pi}}\int_{-\infty}^{\infty}t^k e^{-\frac{t^2}{2}}dt.$$

When k is an odd number, the product function is an odd function and the integral value is 0, we have

$$E\left\{\left[X-E(X)\right]^k\right\}=0,k=1,3,5,\cdots.$$

When k is an even number, the product function is an even function,

$$E\left\{\left[X-E(X)\right]^k\right\}=\frac{2\sigma^k}{\sqrt{2\pi}}\int_{0}^{\infty}t^k e^{-\frac{t^2}{2}}dt,$$

Make a variable substitution $\frac{t^2}{2}=\mu$, we have

$$E\left\{\left[X-E(X)\right]^k\right\}=\frac{2^{\frac{k}{2}}\sigma^k}{\sqrt{\pi}}\int_{0}^{\infty}\mu^{\frac{k-1}{2}}e^{-\mu}d\mu$$
$$=\frac{2^{\frac{k}{2}}\sigma^k}{\sqrt{\pi}}\Gamma\left(\frac{k-1}{2}\right)=(k-1)!!\sigma^k,k=2,4,6,\cdots$$

3) Application of Micro Incremental Method in Probability Theory and Mathematical Statistics

If the probability of an event is a function and depends only on one variable, the equation is found through the micro-increment and the unknown function is obtained by solving the differential equation. This method is called micro-incremental method. Here is a concrete example.

The probability that a machine will stop working due to failure in Δt time is $\alpha\Delta t+o(\Delta t)$ (α is a normal number), assume that the events in which the machine stops working within a non-overlapping time are independent of each other. We know that the machine is working normally at time t_0, and we try to find the probability $P(t)$ that the machine will work normally from time t_0 to time t_0+t.

Since the machine is working normally within $[t_0,t_0+t+\Delta t]$. if and only if the machine is working normally during the $[t_0,t_0+t]$ and $[t_0+t,t_0+t+\Delta t]$ periods. From the hypothesis of the title, it can be seen that the two events are independent of each other, so there are:

$$P(t+\Delta t)=P(t)+P(\Delta t)=P(t)\left[1-\alpha\Delta t-o(\Delta t)\right].$$

So,

$$P(t+\Delta t)-P(t)=-\alpha P(t)\Delta t-P(t)o(\Delta t),$$

As $P(t)\le 1$,and $\Delta t\to 0$, there are:

$$\lim_{\Delta t\to 0}\frac{P(t+\Delta t)-P(t)}{\Delta t}=-\alpha P(t),$$

which means $\frac{dP(t)}{dt}=-\alpha P(t)$,

Solution differential equations,

$P(t)=Ce^{-\alpha t}$, （C is any constant）.

Substituting the initial condition $P(0)=1$ into the above formula, there are $C=1$, so

$P(t)=e^{-\alpha t}$.

The probability of the machine working normally from time t_0 to t_0+t is $e^{-\alpha t}$.

This example is a problem of probability solving in probability theory and mathematical statistics, but it is transformed into a micro-incremental problem in calculus.

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Finally, the probability of an event is obtained by solving a differential equation.

C. Conclusion

Calculus has a history of hundreds of years and the knowledge system is perfect. This is one of the reasons why mathematicians use calculus to solve the problems of probability theory and mathematical statistics. Calculus thought has indeed promoted the rapid development of probability theory and mathematical statistics. At the same time, many ideas of probability theory and mathematical statistics have also been used to solve complex calculus problems. In the future, more advantages of the combination of the two may be found to promote the further development of mathematics.

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