ISSN (ONLINE): 2454-9762
ISSN (PRINT): 2454-9762
Available online at www.ijarmate.com

International Journal of Advanced Research in Management, Architecture, Technology and Engineering (IJARMATE)
Vol. 3, Issue 12, December 2017

A Seventh-Order Convergent Newton-Type Iterative Method for Solving Nonlinear Equations

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***Abstract*—In this paper, we present a modified Seventh-order convergent Newton-type method to solve nonlinear equations. It is free from second derivatives. It requires three evaluations of the functions and two evaluations of derivatives at each iteration. Thus the efficiency index of the presented method is 1.4758 which is better than that of classical Newton's method 1.4142.**

***Key words*—Nonlinear equations; Iterative method; Order of convergence; Newton's method; Efficiency index.**

I. Introduction

In this paper, we consider iterative methods to find a simple root x^* of a nonlinear equation $f(x)=0$, where $f: D \subseteq R \to R$ for an open interval D is a scalar function and it is sufficiently smooth in a neighborhood of x^*.

It is well known that classical Newton's method is one of the basic and important methods for solving non-linear equations by the following iterative scheme [1]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which is quadratically convergent in a neighborhood of x^*.

In past decades, much attention has been paid to develop iterative methods for solving nonlinear equations, and many iterative methods have been developed (see references (see [2]-[10] for more details).

In this paper, we present a modified Newton-type iterative method with order of convergence seven for solving nonlinear equations. The method is free from second derivatives. At each iteration, it requires three evaluations of the functions and two evaluations of derivatives. The efficiency index of the presented method is better than that of classical Newton's method.

II. THE MODIFIED METHOD AND ITS CONVERGENCE

Let us consider the following iterative algorithm.

Algorithm 1. For given x_0, we consider the following iteration method for solving nonlinear equation (1)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

$$z_n = y_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'(x_n)^2}, \tag{3}$$

$$x_{n+1} = z_n + \frac{2f'(x_n)^2}{f'(x_n)^2 - 4f'(x_n)f'(y_n) + f'(y_n)^2}\frac{f(z_n)}{f'(x_n)}. \tag{4}$$

For Algorithm 1, we have the following convergence result.

THEOREM 1. Assume that the function $f: D \subseteq R \to R$ has a single root $x^* \in D$, where D is an open interval. If $f(x)$ has first, second and third derivatives in the interval D, then Algorithm 1 defined by (2)-(4) is seventh-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1} = 10c_2^4(2c_2^2 - c_3)e_n^7 + O(e_n^8) \tag{5}$$

where

$$e_n = x_n - x^*,\ c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, k = 1,2,\cdots.$$

Proof. Let x^* be the simple root of $f(x)$, $c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}, k = 1,2,\cdots$, and $e_n = x_n - x^*$. Consider the iteration function $F(x)$ defined by

$$F(x) = z(x) + \frac{2f'(x)^2}{f'(x)^2 - 4f'(x)f'(y(x)) + f'(y(x))^2}\frac{f(z(x))}{f'(x)} \tag{6}$$

where

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Vol. 3, Issue 12, December 2017

$$z(x) = y(x) - \frac{f(x)}{f'(x)} - \frac{2f(y(x))}{f'(x)} + \frac{f(y(x))f'(y(x))}{f'(x)^2},$$

$$y(x) = x - \frac{f(x)}{f'(x)}.$$

By some computations using Maple we can obtain

$$F(x^*) = x^*,\ F^{(i)}(x^*) = 0,\ i = 1,2,3,4,5,6\ ,$$

$$F^{(7)}(x^*) = -\frac{525 f^{(2)}(x^*)^4[f'(x^*)f^{(3)}(x^*) - 3f''(x^*)^2]}{f'(x^*)^6}. \quad (7)$$

Furthermore, from the Taylor expansion of $F(x_n)$ at x^*, we get

$$x_{n+1} = F(x_n) = F(x^*) + \sum_{k=1}^{7} \frac{F^{(k)}(x^*)}{k!}(x_n - x^*)^k + O((x_n - x^*)^8). \quad (8)$$

Substituting (7) into (8) yields

$$x_{n+1} = x^* + e_{n+1} = x^* + 10c_2^4(2c_2^2 - c_3)e_n^7 + O(e_n^8).$$

Therefore, we have

$$e_{n+1} = 10c_2^4(2c_2^2 - c_3)e_n^7 + O(e_n^8),$$

which means the order of convergence of the Algorithm 1 is seven. The proof is completed.

Now, we consider efficiency index defined as $p^{1/\omega}$, where p is the order of the method and ω is the number of function evaluations per iteration required by the method. It is not hard to see that the efficiency index of Algorithm 1 is 1.4758 which is better than that of classical Newton's method (NM) 1.4142.

III. CONCLUSIONS

In this paper, we present and analyze a modified seventh-order convergent Newton-type iterative method for solving nonlinear equations. It is free from second derivatives. It requires three evaluations of the functions and two evaluations of derivatives in each step. The seventh-order method proposed in this paper is more efficient than Newton's method.

ACKNOWLEDGMENT

The work is supported by Project of Natural Science Foundation of Shandong province (ZR2016AM06), Excellent Young Scientist Foundation of Shandong Province (BS2011SF024).

REFERENCES

[1] J. F. Traub, *Iterative Methods for Solution of Equations*, Prentice-Hall, Englewood Clis, NJ(1964).
[2] M.A. Noor, and K. I. Noor, "Fifth-order iterative methods for solving nonlinear equations", *Appl. Math. Comput.*, vol. 188, no. 1, pp. 406-410, 2007.
[3] J. Kou, "The improvements of modified Newton's method", *Appl. Math. Comput.*, vol. 189, no. 1, pp. 602-609, 2007.
[4] L. Fang, and G. He, "Some modifications of Newton's method with higher-order convergence for solving nonlinear equations", *J. Comput. Appl. Math.*, vol. 228, no. 1, pp. 296-303, 2009.
[5] F.A. Potra, and Potra-Pták, "Nondiscrete induction and iterative processes", *Research Notes in Mathematics*,103(ISBN-10: 0273086278) (1-250), Pitman, Boston, 1984.
[6] J. Kou, Y. Li, and X. Wang, "Some variants of Ostrowski's method with seventh-order convergence", *J. Comput. Appl. Math.,* vol. 209, no. 2, pp. 153-159, 2007.
[7] C. Chun, "Construction of Newton-like iteration methods for solving nonlinear equations". *Numer. Math.*, vol. 104, no. 3, pp. 297-315, 2006.
[8] A. Galantai, "The theory of Newton's method", *J. Comput. Appl. Math.*, vol. 124, pp. 25-44, 2000.
[9] C. Chun, "Some fourth-order iterative methods for solvingnonlinear equations", *Applied Mathematics and Computation*, vol. 195, pp. 454-459, 2008.
[10] S. Huang, and Z. Wan, "A new nonmonotone spectral residual method for nonsmooth nonlinear equations", *Journal of Computational and Applied Mathematics*, vol. 313, pp. 82-101, 2017.

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